RESOLUTION OF THE WAVEFRONT SET USING CONTINUOUS SHEARLETS

GITTA KUTYNIOK AND DEMETRIO LABATE

Abstract. It is known that the continuous wavelet transform of a function f decays very rapidly near the points where f is smooth, while it decays slowly near the irregular points. This property allows one to precisely identify the singular support of f. However, the continuous wavelet transform is unable to provide additional information about the geometry of the singular points. In this paper, we introduce a new transform for functions and distributions on $\mathbb{R}^2$, called the Continuous Shearlet Transform. This is defined by $\mathcal{SH}_f(a,s,t) = \langle f, \psi_{ast}\rangle$, where the analyzing elements ψ_{ast} are dilated and translated copies of a single generating function ψ and, thus, they form an affine system. The resulting continuous shearlets ψ_{ast} are smooth functions at continuous scales $a > 0$, locations $t \in \mathbb{R}^2$ and oriented along lines of slope $s \in \mathbb{R}$ in the frequency domain. The Continuous Shearlet Transform transform is able to identify not only the location of the singular points of a distribution f, but also the orientation of their distributed singularities. As a result, we can use this transform to exactly characterize the wavefront set of f.

1. Introduction

It is well-known that if ψ is a 'nice' continuous wavelet on $\mathbb{R}^n$, and f is a function that is smooth apart from a discontinuity at $x_0 \in \mathbb{R}^n$, then the continuous wavelet transform

$$\mathcal{W}_f(a,t) = a^{-\frac{n}{2}} \int_{\mathbb{R}} f(x)\, \psi\left(a^{-1}(x-t)\right)\, dx, \quad a > 0, t \in \mathbb{R}^n$$

decays rapidly as $a \to 0$, unless t is near x_0 [16, 21]. As a consequence, the continuous wavelet transform is able to resolve the *singular support* of the distribution f, that is, to identify the set of points where f is not regular. However, the transform $\mathcal{W}_f(a,t)$ is unable to provide additional information about the *geometry* of the set of singularities of a more general distribution f. For example, in many situations one would like to identify not only the location of a certain distributed singularity, but also its orientation. This is very relevant, in particular, in the study of the propagation of singularities associated with partial differential equations [17, 23].

In this paper, we introduce a new two-dimensional continuous transform, called *continuous shearlet transform*, that is mapping a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$ to $\mathcal{SH}_f(a,s,t)$, where $a > 0$, $s \in \mathbb{R}$ and $t \in \mathbb{R}^2$, are the scale, shear and location variables, respectively. The transform is defined by $\mathcal{SH}_f(a,s,t) = \langle f, \psi_{ast}\rangle$, where the analyzing elements ψ_{ast}, called *continuous shearlets*, are dilated and

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translated copies of a single generating function ψ, thereby constituting a continuous 2-dimensional wavelet-like system. The generator $\psi \in L^2(\mathbb{R}^2)$ is chosen to be arbitrarily smooth and has compact support in the frequency domain with a needlelike structure to capture directionality. For the dilation matrix we employ the product of a parabolic scaling matrix associated with $a > 0$ and a shear matrix associated with $s \in \mathbb{R}$, whereas $t \in \mathbb{R}^2$ serves as the translation parameter. The system $\{\psi_{ast} : a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2\}$ will be shown to satisfy a Calderòn-type formula with respect to a special measure, that is, it forms a reproducing system for $L^2(\mathbb{R}^2)$.

We will prove that the continuous shearlet transform is able to identify not only the singular support of the distribution f, but also the orientation of distributed singularities along curves. In particular, the decay properties of the continuous shearlet transform as $a \to 0$ precisely characterize the *wavefront set* of f (see Section 5).

Historically, the idea of using wavelet-like transforms to identify the set of singularities of a distribution can be traced back to the *wave packet transform*, introduced independently by Bros and Iagolnitzer [1] and Córdoba and Fefferman [10]. More recently, Smith [22] and Candès and Donoho [6, 7] have introduced a continuous transform that uses parabolic scaling in polar coordinates and has the ability to identify the wavefront set of a distribution. In particular, the Continuous Curvelet Transform (CCT) of Candès and Donoho has properties similar to the continuous shearlet transform presented in this paper. Namely, in both cases, the wavefront set is exactly characterized by the decay properties of the continuous transform. Unlike the CCT, however, our approach exploits the framework of *affine systems*. As a consequence, the continuous shearlet transform is not only much closer in spirit to the traditional continuous wavelet transform, but also avoids the more complicated structure of the curvelet construction, which uses infinitely many different generators.

Another motivation for this investigation and the use of the framework of affine systems comes from the study of discrete wavelets, and, more specifically their ability to approximate efficiently smooth functions with singularities. This property is closely related to the micro-localization property of the continuous wavelet transform. To illustrate this point, let f be a one-dimensional function that is smooth apart from a discontinuity at x_0 and consider the wavelet representation of f:

$$f = \sum_{k,j \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x),$$

where $\psi_{j,k}(x) = 2^{-j/2}\, \psi(2^{-j}x - k)$ and ψ is a 'nice' wavelet. Since the wavelet transform $\mathcal{W}_f(2^{-j}, k) = \langle f, \psi_{j,k} \rangle$ decays rapidly for $j \to \infty$ unless k is near x_0, it follows that very few coefficients of the wavelet representation are sufficient to approximate f accurately. Indeed, the wavelet representation is optimally sparse for this type of functions (cf. [20, Ch.9]). Therefore, the properties of the continuous wavelet transform are crucial in order to achieve a sparse representation.

Traditional wavelets, however, have a very limited capability in dealing with discontinuities in higher dimensions. Consider, for example, the wavelet representation of a 2-dimensional function that is smooth away from a discontinuity along a curve. Because the discontinuity is spatially distributed, it interacts extensively with the elements of the wavelet basis, and, thus, many wavelet coefficients are

needed to represent f accurately. Related to this is the fact that the corresponding two-dimensional continuous wavelet transform is unable to 'track' the discontinuous curve. As pointed out by several authors (cf. [4, 5]), in order to represent multidimensional functions efficiently, one has to use representations that are much more flexible than traditional wavelets, and have the ability to capture the geometry of multidimensional phenomena. One goal of this paper is to show that the study of the continuous wavelet transform associated with the affine group on $\mathbb{R}^2$ provides a unifying perspective on the construction of such representations.

The study of the discrete analog of the continuous shearlet transform is currently being developed by the authors and their collaborators [12, 13, 14, 15, 18]. In particular, it was shown that, thanks to the mathematical structure of affine systems, discrete shearlets are associated with a Multiresolution analysis similar to traditional wavelets, which is very relevant for the development of fast algorithmic implementations. In addition, shearlets provide optimally sparse representations for functions in certain classes and can be easily generalized to higher dimensions.

The paper is organized as follows. In Section 2 we recall the basic properties of affine systems on $\mathbb{R}^n$ and the continuous wavelet transform, before introducing the continuous shearlet transform in Section 2. In Section 3 we apply this new transform to several examples of distributions containing different types of singularities. The main result of this paper is proved in Section 5, where we show that the continuous shearlet transform exactly characterizes the wavefront set of a distribution. Finally, in Section 6, we discuss several variants and generalizations of our construction.

2. Affine systems and wavelets

2.1. **One-dimensional Continuous Wavelet Transform.** Let $\mathbb{A}_1$ be the *affine group* associated with $\mathbb{R}$, consisting of all pairs (a,t), $a,t \in \mathbb{R}, a > 0$, with group operation $(a,t)\cdot(a',t') = (aa', t+at')$. The *(continuous) affine systems* generated by $\psi \in L^2(\mathbb{R})$ are obtained from the action of the *quasi–regular representation* $\pi_{(a,t)}$ of $\mathbb{A}_1$ on $L^2(\mathbb{R})$, that is

$$\left\{\psi_{a,t}(x) = \pi_{(a,t)}\,\psi(x) = T_t\,D_a\,\psi(x) : \,(a,t)\in\mathbb{A}_1\right\},$$

where the *translation operator* T_t is defined by $T_t\psi(x) = \psi(x-t)$ and the *dilation operator* D_a is defined by $D_a\psi(x) = a^{-1/2}\psi(a^{-1}x)$.

It was observed by Calderòn [2] that, if ψ satisfies the *admissibility* condition

$$\int_0^\infty |\hat\psi(a\xi)|^2\,\frac{da}{a} = 1 \quad \text{for a.e. } \xi\in\mathbb{R}, \tag{2.1}$$

then any $f \in L^2(\mathbb{R})$ can be recovered via the reproducing formula:

$$f = \int_{\mathbb{A}_1} \langle f, \psi_{a,t}\rangle\,\psi_{a,t}\,d\mu(a,t),$$

where $d\mu(a,t) = dt\,\frac{da}{a^2}$ is the left Haar measure of $\mathbb{A}_1$. Here the Fourier transform is defined by $\hat\psi(\xi) = \int \psi(x)\,e^{-2\pi i\xi x}\,dx$. As usual, $\check\psi$ will denote the inverse Fourier transform. The function ψ is called a *continuous wavelet*, if ψ satisfies (2.1), and $\mathcal{W}_f(a,t) = \langle f, \psi_{a,t}\rangle$ is the *continuous wavelet transform* of f. We refer to [11] for more details about this.

Discrete affine systems and wavelets are obtained by 'discretizing' appropriately the corresponding continuous systems. In fact, by replacing $(a,t)\in\mathbb{A}_1$ with the

discrete set $(2^j, 2^j m)$, $j, m \in \mathbb{Z}$, one obtains the discrete dyadic affine system

$$\big\{\psi_{j,m}(x) = T_{2^j m}\, D_2^j\, \psi(x) = D_2^j\, T_m\, \psi(x) :\ j, m \in \mathbb{Z}\big\}, \tag{2.2}$$

and ψ is called a *wavelet* if (2.2) is an orthonormal basis or, more generally, a Parseval frame for $L^2(\mathbb{R})$.

Recall that a countable collection $\{\psi_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is a *Parseval frame* (sometimes called a *tight frame*) for $\mathcal{H}$, if $\sum_{i\in I} |\langle f, \psi_i\rangle|^2 = \|f\|^2$ for all $f \in \mathcal{H}$. This is equivalent to the reproducing formula $f = \sum_{i\in I} \langle f, \psi_i\rangle\, \psi_i$ for all $f \in \mathcal{H}$, where the series converges in the norm of $\mathcal{H}$. Thus Parseval frames provide basis-like representations even though a Parseval frame need not be a basis in general. We refer the reader to [8, 9] for more details about frames.

2.2. **Higher-dimensional Continuous Wavelet Transform.** The natural way of extending the theory of affine systems to higher dimensions is by replacing $\mathbb{A}_1$ with the *full affine group of motions on* $\mathbb{R}^n$, $\mathbb{A}_n$, consisting of the pairs $(M, t) \in GL_n(\mathbb{R}) \times \mathbb{R}^n$ with group operation $(M, t)\cdot(M', t') = (MM', t + Mt')$. Similarly to the one-dimensional case, the affine systems generated by $\psi \in L^2(\mathbb{R}^n)$ are given by

$$\big\{\psi_{M,t}(x) = T_t\, D_M\, \psi(x) :\ (M, t) \in \mathbb{A}_n\big\},$$

where here the *dilation operator* D_M is defined by $D_M\, \psi(x) = |\det M|^{-\frac{1}{2}}\, \psi(M^{-1}x)$. The generalization of the Calderòn admissibility condition to higher dimensions and the construction of multidimensional wavelets is a far more complex task than the corresponding one-dimensional problem, and yet not fully understood. We refer to [19, 25] for more details.

Now let G be a subset of $GL_n(\mathbb{R})$ and define $\Lambda \subseteq \mathbb{A}_n$ by $\Lambda = \{(M, t) : M \in G, t \in \mathbb{R}^n\}$. If there exists a function $\psi \in L^2(\mathbb{R}^n)$ such that, for all $f \in L^2(\mathbb{R}^n)$, we have:

$$f = \int_{\mathbb{R}^n} \int_G \langle f, T_t\, D_M\, \psi\rangle\, T_t\, D_M\, \psi\, d\lambda(M)\, dt, \tag{2.3}$$

where λ is a measure on G, then ψ is a *continuous wavelet* with respect to Λ. The following result, that is a simple modification of Theorem 2.1 in [25], gives an exact characterization of all those $\psi \in L^2(\mathbb{R}^n)$ that are continuous wavelets with respect to Λ. The proof of this theorem is reported in the Appendix.

Theorem 2.1. *Equality (2.3) is valid for all $f \in L^2(\mathbb{R}^n)$ if and only if, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,*

$$\Delta(\psi)(\xi) = \int_G |\hat{\psi}(M^t \xi)|^2\, |\det M|\, d\lambda(M) = 1. \tag{2.4}$$

The choice of the measure λ on G is not unique. If G is not simply a subset of $GL_n(\mathbb{R})$, but also a subgroup, then we can use the left Haar measure on G which is unique up to a multiplicative constant. Also, observe that Theorem 2.1 extends to functions on subspaces of $L^2(\mathbb{R}^n)$ of the form

$$L^2(V)^\vee = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset V\}.$$

2.3. **Localization of Wavelets.** The decay properties of the functions $\psi_{M,t} = T_t\, D_M\, \psi$, where $\hat{\psi} \in C_0^\infty$, are described by the following proposition.

Proposition 2.2. *Suppose that $\psi \in L^2(\mathbb{R}^n)$ is such that $\hat{\psi} \in C_0^\infty(R)$, where $R = supp\,\hat{\psi} \subset \widehat{\mathbb{R}}^n$. Then, for each $k \in \mathbb{N}$, there is a constant C_k such that, for any $x \in \mathbb{R}^n$, we have*

$$|\psi_{M,t}(x)| \le C_k \,|\det M|^{-\frac{1}{2}}\,(1+|M^{-1}(x-t)|^2)^{-k}.$$

In particular, $C_k = k\,m(R)\left(\|\hat{\psi}\|_\infty + \|\triangle^k\hat{\psi}\|_\infty\right)$, where $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial\xi_i^2}$ is the frequency domain Laplacian operator and $m(R)$ is the Lebesgue measure of R.

The proof of this proposition relies on the following known observation, whose proof is included for completeness.

Lemma 2.3. *Let g be such that $\hat{g} \in C_0^\infty(R)$, where $R \subset \mathbb{R}^n$ is the $supp\,\hat{g}$. Then, for each $k \in \mathbb{N}$, there is a constant C_k such that for any $x \in \mathbb{R}^n$*

$$|g(x)| \le C_k\,(1+|x|^2)^{-k}.$$

In particular, $C_k = k\,m(R)\left(\|\hat{g}\|_\infty + \|\triangle^k\hat{g}\|_\infty\right)$.

Proof. Since $g(x) = \int_R \hat{g}(\xi)\,e^{2\pi i\xi x}\,d\xi$, then, for every $x \in \mathbb{R}^2$,

$$|g(x)| \le m(R)\,\|\hat{g}\|_\infty. \tag{2.5}$$

An integration by parts shows that

$$\int_R \triangle\hat{g}(\xi)\,e^{2\pi i\xi x}\,d\xi = -(2\pi)^2\,|x|^2\,g(x)$$

and thus, for every $x \in \mathbb{R}^2$,

$$(2\pi\,|x|)^{2k}\,|g(x)| \le m(R)\,\|\triangle^k\hat{g}\|_\infty. \tag{2.6}$$

Using (2.5) and (2.6), we have

$$\left(1+(2\pi\,|x|)^{2k}\right)|g(x)| \le m(R)\left(\|\hat{g}\|_\infty + \|\triangle^k\hat{g}\|_\infty\right). \tag{2.7}$$

Observe that, for each $k \in \mathbb{N}$,

$$(1+|x|^2)^k \le \left(1+(2\pi)^2\,|x|^2\right)^k \le k\left(1+(2\pi\,|x|)^{2k}\right).$$

Using this last inequality and (2.7), we have that for each $x \in \mathbb{R}^n$

$$|g(x)| \le k\,m(R)\,(1+|x|^2)^{-k}\left(\|\hat{g}\|_\infty + \|\triangle^k\hat{g}\|_\infty\right). \quad \square$$

A simple re-scaling argument now proves Proposition 2.2.

Proof of Proposition 2.2. A direct computation gives:

$$\begin{aligned}\psi(M^{-1}(x-t)) &= \int_R \hat{\psi}(\xi)\,e^{2\pi i M^{-1}(x-t)\xi}\,d\xi \\ &= \int_R \hat{\psi}(\xi)\,e^{2\pi i (x-t)M^{-t}\xi}\,d\xi \\ &= \int_{RM^{-1}} \hat{\psi}(M^t\eta)\,e^{2\pi i(x-t)\eta}\,|\det M|\,d\eta.\end{aligned}$$

It follows that

$$|\psi(M^{-1}(x-t))| \le m(M^{-1}R)\,|\det M|\,\|\hat{\psi}(M^t\cdot)\|_\infty = m(R)\,\|\hat{\psi}\|_\infty.$$

Using a simple modification of the argument in Lemma 2.3, we have that

$$(2\pi\,|M^{-1}(x-t)|)^{2k}\,|\psi(M^{-1}(x-t))| \le m(R)\,\|\triangle^k\hat{\psi}\|_\infty.$$

Next, arguing again as in Lemma 2.3 we have that

$$|\psi(M^{-1}(x-t))| \le k\, m(R)\,(1+|M^{-1}(x-t)|^2)^{-k}\left(\|\hat\psi\|_\infty + \|\triangle^k \hat\psi\|_\infty\right).$$

This completes the proof. $\Box$

3. Continuous Shearlet Transform

3.1. **Definition.** In this paper, we will be interested in the affine systems obtained when Λ is a subset of $\mathbb{A}_2$ of the form

$$\Lambda = \{(M,t) :\, M \in G, t \in \mathbb{R}^2\}, \tag{3.1}$$

and $G \subset GL_2(\mathbb{R})$ is the set of matrices:

$$G = \left\{ M = M_{as} = \begin{pmatrix} a & -\sqrt{a}\, s \\ & \\ 0 & \sqrt{a} \end{pmatrix}, \quad a \in I,\, s \in S \right\}, \tag{3.2}$$

where $I \subset \mathbb{R}^+$, $S \subset \mathbb{R}$. It is useful to notice that the matrices M can be factorized as $M = B\,A$, where B is the *shear matrix* $B = \left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right)$ and A is the diagonal matrix $A = \left(\begin{smallmatrix} a & 0 \\ 0 & \sqrt{a} \end{smallmatrix}\right)$. In particular, A produces *parabolic scaling*, that is, $f(Ax) = f\left(A \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right)$ leaves invariant the parabola $x_1 = x_2^2$. Thus, the matrix M can be interpreted as the superposition of parabolic scaling and shear transformation.

We will now construct a continuous wavelet on $\mathbb{R}^2$ associated with the subset Λ of the affine group, given by (3.1). We will consider two situations, corresponding to $I = \mathbb{R}^+$, $S = \mathbb{R}$ or $I = \{a : 0 \le a \le 1\}$, $S = \{s \in \mathbb{R} : |s| \le s_0\}$, for some $s_0 > 0$.

For $\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2$, $\xi_2 \ne 0$, let ψ be given by

$$\hat\psi(\xi) = \hat\psi(\xi_1,\xi_2) = \hat\psi_1(\xi_1)\,\hat\psi_2(\tfrac{\xi_2}{\xi_1}). \tag{3.3}$$

Proposition 3.1. *Let Λ be given by (3.1) and (3.2) with $I = \mathbb{R}^+$, $S = \mathbb{R}$, and $\psi \in L^2(\mathbb{R}^2)$ be given by (3.3) where:*

(i) *$\psi_1 \in L^2(\mathbb{R})$ satisfies the Calderòn condition (2.1);*
(ii) *$\|\psi_2\|_{L^2} = 1$.*

Then ψ is a continuous wavelet for $L^2(\mathbb{R}^2)$ with respect to Λ.

Proof. A direct computation shows that $M^t(\xi_1,\xi_2)^t = (a\xi_1, a^{1/2}(\xi_2 - s\xi_1))^t$. By choosing as measure $d\lambda(M) = \frac{da}{|\det M|^2}\, ds$, the admissibility condition (2.4) becomes

$$\Delta(\psi)(\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}^+} |\hat\psi_1(a\,\xi_1)|^2\,|\hat\psi_2(a^{-\frac12}(\tfrac{\xi_2}{\xi_1} - s))|^2\; a^{-\frac32} da\, ds = 1. \tag{3.4}$$

Thus, by Theorem 2.1, to show that ψ is a continuous wavelet it is sufficient to show that (3.4) is satisfied. Using the assumption on ψ_1 and ψ_2, we have:

$$\begin{aligned}
\Delta(\psi)(\xi) &= \int_{\mathbb{R}}\int_{\mathbb{R}^+} |\hat\psi_1(a\,\xi_1)|^2\,|\hat\psi_2(a^{-\frac12}(\tfrac{\xi_2}{\xi_1} - s))|^2\; a^{-\frac32} da\, ds \\
&= \int_{\mathbb{R}^+} |\hat\psi_1(a\,\xi_1)|^2 \left(\int_{\mathbb{R}} |\hat\psi_2(a^{-\frac12}\,\tfrac{\xi_2}{\xi_1} - s)|^2\; ds\right) \frac{da}{a} \\
&= \int_{\mathbb{R}^+} |\hat\psi_1(a\,\xi_1)|^2\, \frac{da}{a} = 1 \quad \text{ for a.e. } \xi = (\xi_1,\xi_2) \in \mathbb{R}^2.
\end{aligned}$$

This shows that equality (3.4) is satisfied and, hence, ψ is a continuous wavelet. $\square$

If the set S is not all of $\mathbb{R}$, then we need some additional assumptions on ψ. Consider the subspace of $L^2(\mathbb{R}^2)$ given by $L^2(C)^\vee = \{f \in L^2(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset C\}$, where

$$C = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geq 2 \text{ and } |\tfrac{\xi_2}{\xi_1}| \leq 1\}.$$

We have the following result.

Proposition 3.2. *Let Λ be given by (3.1) and (3.2) with $I = \{a : 0 \leq a \leq 1\}$, $S = \{s \in \mathbb{R} : |s| \leq 2\}$, and $\psi \in L^2(\mathbb{R}^2)$ be given by (3.3) where:*

(i) *$\psi_1 \in L^2(\mathbb{R})$ satisfies the Calderòn condition (2.1), and supp $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$;*

(ii) *$\|\psi_2\|_{L^2} = 1$ and supp $\hat{\psi}_2 \subset [-1, 1]$.*

Then ψ is a continuous wavelet for $L^2(C)^\vee$ with respect to Λ, that is, for all $f \in L^2(C)^\vee$,

$$f(x) = \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 \langle f, \psi_{ast} \rangle \, \psi_{ast}(x) \, \frac{da}{a^3} \, ds \, dt.$$

Proof. We apply again Theorem 2.1, for functions on $L^2(C)^\vee$. Using the assumptions on ψ_2, S and I we have that, for $\xi \in C$:

$$\int_{\frac{1}{\sqrt{a}}(\frac{\xi_2}{\xi_1}-2)}^{\frac{1}{\sqrt{a}}(\frac{\xi_2}{\xi_1}+2)} |\hat{\psi}_2(s)|^2 \, ds = \int_{-1}^{1} |\hat{\psi}_2(s)|^2 \, ds = 1.$$

Thus, for a.e. $\xi \in C$ we have that

$$\begin{aligned}
\Delta(\psi)(\xi) &= \int_{-2}^{2} \int_0^1 |\hat{\psi}(M_{as}^t \xi)|^2 \, a^{-\frac{3}{2}} \, da \, ds \\
&= \int_{-2}^{2} \int_0^1 |\hat{\psi}_1(a\xi_1)|^2 \, |\hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1} - s))|^2 a^{-\frac{3}{2}} \, da \, ds \\
&= \int_0^1 |\hat{\psi}_1(a\xi_1)|^2 \int_{\frac{1}{\sqrt{a}}(\frac{\xi_2}{\xi_1}-2)}^{\frac{1}{\sqrt{a}}(\frac{\xi_2}{\xi_1}+2)} |\hat{\psi}_2(s)|^2 \, ds \, \frac{da}{a} \\
&= \int_0^1 |\hat{\psi}_1(a\xi_1)|^2 \, \frac{da}{a}.
\end{aligned}$$

Since $\xi_1 \geq 2$, using the assumptions on the support of $\hat{\psi}_1$ and condition (2.1), from the last expression we have that, for a.e. $\xi \in C$,

$$\Delta(\psi)(\xi) = \int_0^{\xi_1} |\hat{\psi}_1(a)|^2 \, \frac{da}{a} = \int_{\frac{1}{2}}^{2} |\hat{\psi}_1(a)|^2 \, \frac{da}{a} = \int_0^{\infty} |\hat{\psi}_1(a)|^2 \, \frac{da}{a} = 1.$$

This shows that the admissibility condition (2.4) for this system is satisfied and this completes the proof. $\square$

There are several examples of functions ψ_1 and ψ_2 satisfying the assumptions of Proposition 3.1 as well as Proposition 3.2. In addition, we can choose ψ_1, ψ_2 such that $\hat{\psi}_1$, $\hat{\psi}_2 \in C_0^\infty$ (see [12, 15] for the construction of these functions).

Now we can define the *continuous shearlet transform:*

Definition 3.3. Let $\psi \in L^2(\mathbb{R}^2)$ be given by (3.3) where:

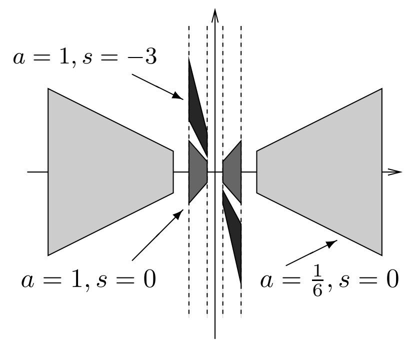

FIGURE 1. Support of the shearlets $\hat{\psi}_{ast}$ (in the frequency domain) for different values of a and s.

(i) $\psi_1 \in L^2(\mathbb{R})$ satisfies the Calderòn condition (2.1), and $\hat{\psi}_1 \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$;

(ii) $\|\psi_2\|_{L^2} = 1$, and $\hat{\psi}_2 \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$ and $\hat{\psi}_2 > 0$ on $(-1, 1)$.

The functions generated by ψ under the action of Λ, namely:

$$\psi_{ast}(x) = T_t\, D_M \psi(x) = a^{-\frac{3}{4}}\, \psi\left(\begin{pmatrix} a & -\sqrt{a}\, s \\ 0 & \sqrt{a} \end{pmatrix}^{-1} (x - t)\right),$$

where $a \in I \subset \mathbb{R}^+$, $s \in S \subset \mathbb{R}$ and $t \in \mathbb{R}^2$, are called *continuous shearlets*. The *continuous shearlet transform* of f is defined by

$$\mathcal{SH}_f(a, s, t) = \langle f, \psi_{ast} \rangle, \quad a \in I \subset \mathbb{R}^+, s \in S \subset \mathbb{R}, t \in \mathbb{R}^2.$$

Observe that, unlike the traditional wavelet transform which depends only on scale and translation, the shearlet transform is a function of three variables, that is, the scale a, the shear s and the translation t. Many properties of the continuous shearlets are more evident in the frequency domain. A direct computation shows that

$$\begin{aligned} \hat{\psi}_{ast}(\xi) &= a^{\frac{3}{4}}\, e^{-2\pi i \xi t}\, \hat{\psi}(a\,\xi_1, \sqrt{a}(\xi_2 - s\,\xi_1)) \\ &= a^{\frac{3}{4}}\, e^{-2\pi i \xi t}\, \hat{\psi}_1(a\,\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1} - s)). \end{aligned}$$

Thus, each function $\hat{\psi}_{ast}$ is supported in the set:

$$\operatorname{supp} \hat{\psi}_{ast} \subset \{(\xi_1, \xi_2) : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}],\ |\tfrac{\xi_2}{\xi_1} - s| \le \sqrt{a}\}.$$

As illustrated in Figure 1, each continuous shearlet ψ_{ast} has frequency support on a pair of trapezoids, symmetric with respect to the origin, oriented along a line of slope s. The support becomes increasingly thin as $a \to 0$.

When $S = \mathbb{R}$ and $I = \mathbb{R}^+$, by Proposition 3.1, the continuous shearlet transform provides a reproducing formula (2.3) for all $f \in L^2(\mathbb{R}^2)$:

$$\|f\|^2 = \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \int_0^\infty |\mathcal{SH}_f(a, s, t)|^2\, \frac{da}{a^3}\, ds\, dt.$$

On the other hand, if S, I are bounded sets, by Proposition 3.2, the continuous shearlet transform provides a reproducing formula only for functions in a proper subspace of $L^2(\mathbb{R}^2)$. However, even when S, I are bounded, it is possible to obtain a reproducing formula for all $f \in L^2(\mathbb{R}^2)$ as follows. Let

$$\hat{\psi}^{(v)}(\xi) = \hat{\psi}^{(v)}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_2)\, \hat{\psi}_2(\tfrac{\xi_1}{\xi_2}),$$

where $\hat{\psi}_1$, $\hat{\psi}_2$ are defined as in Definition 3.3, and let $\Lambda^{(v)} = \{(M,t) : M \in G^{(v)}, t \in \mathbb{R}^2\}$, where

$$G^{(v)} = \left\{ M = M_{as} = \begin{pmatrix} \sqrt{a} & 0 \\ -\sqrt{a}\, s & a \end{pmatrix}, \quad a \in I,\, s \in S \right\}, \tag{3.5}$$

Then, proceeding as above, it is easy to show that $\psi^{(v)}$ is a continuous wavelet for $L^2(C^{(v)})^\vee$ with respect to $\Lambda^{(v)}$, where $C^{(v)}$ is the vertical cone:

$$C^{(v)} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \geq 2 \text{ and } |\tfrac{\xi_2}{\xi_1}| > 1\}.$$

Accordingly, we introduce the shearlets $\psi_{ast}^{(v)} = T_t\, D_M \psi^{(v)}$, for $(M,t) \in \Lambda^{(v)}$, and the associated continuous shearlet transform $\mathcal{SH}_f^{(v)}(a,s,t) = \langle f, \psi_{ast}^{(v)} \rangle$. Finally, let $W(x)$ be such that $\hat{W}(\xi) \in C^\infty(\mathbb{R}^2)$ and

$$|\hat{W}(\xi)|^2 + \chi_{C_1}(\xi) \int_0^1 |\hat{\psi}_1(a\xi_1)|^2 \, \frac{da}{a} + \chi_{C_2}(\xi) \int_0^1 |\hat{\psi}_1(a\xi_2)|^2 \, \frac{da}{a} = 1, \tag{3.6}$$

for a. e. $\xi \in \mathbb{R}^2$, where $C_1 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| \leq 1\}$, $C_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| > 1\}$. Then it follows that W is a C^∞ window function in $\mathbb{R}^2$ with $\hat{W}(\xi) = 1$ for $\xi \in [-1/2, 1/2]^2$, $\hat{W}(\xi) = 0$ outside the box $\{\xi \in [-2,2]^2\}$. Finally, let $(P_{C_1} f)^\wedge = \hat{f}\, \chi_{C_1}$ and $(P_{C_2} f)^\wedge = \hat{f}\, \chi_{C_2}$. Then, for each $f \in L^2(\mathbb{R}^2)$ we have:

$$\begin{aligned} \|f\|^2 &= \int_{\mathbb{R}} |\langle f, T_t\, W \rangle|^2 \, dt + \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 |\mathcal{SH}_{(P_{C_1} f)}(a,s,t)|^2 \, \frac{da}{a^3} \, ds \, dt \\ &+ \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 |\mathcal{SH}^{(v)}_{(P_{C_2} f)}(a,s,t)|^2 \, \frac{da}{a^3} \, ds \, dt. \end{aligned} \tag{3.7}$$

The proof of this equality is reported in the Appendix. Equation (3.7) shows that f is continuously reproduced by using isotropic window functions at coarse scales, and two sets of continuous shearlets at fine scales: one set corresponding to the horizontal cone C (in the frequency domain) and another set corresponding to the vertical cone $C^{(v)}$. The advantage of this construction, with respect to the simpler one where $S = \mathbb{R}$, is that in this case the set S associated with the shear variable is the closed interval $S = \{s : |s| \leq 2\}$. This property will be important in Subsection 4.4 and Section 5.

There are other choices of the subset Λ, given by (3.1), generating affine systems with properties similar to the continuous shearlets. Variants and generalizations of this construction will be discussed in Section 6.

3.2. **Localization of Shearlets.** Since the continuous wavelet ψ associated with the continuous shearlet transform satisfies $\hat{\psi} \in C_0^\infty(\widehat{\mathbb{R}}^2)$, it follows that the continuous shearlets *decay rapidly* as $|x| \to \infty$, that is:

$$\psi_{ast}(x) = O(|x|^{-k}) \quad \text{as } |x| \to \infty, \quad \text{for every } k \geq 0.$$

More precisely, we have the following result.

Proposition 3.4. *Let $\psi \in L^2(\mathbb{R}^2)$ be the continuous wavelet associated with the continuous shearlet transform, and let M be defined as in (3.2). Then, for each $k \in \mathbb{N}$, there is a constant C_k such that, for any $x \in \mathbb{R}^2$, we have*

$$\begin{aligned} |\psi_{ast}(x)| &\leq C_k \, |\det M|^{-\frac{1}{2}} \, (1 + |M^{-1}(x-t)|^2)^{-k} \\ &= C_k \, a^{-\frac{3}{4}} \, (1 + a^{-2}(x_1 - t_1)^2 + 2a^{-2}s(x_1-t_1)(x_2-t_2) \\ &\quad + a^{-1}(1 + a^{-1}s^2)(x_2-t_2)^2)^{-k}. \end{aligned}$$

In particular, $C_k = k \, \frac{15}{2} \frac{\sqrt{a}+s}{a^2} \left(\|\hat{\psi}\|_\infty + \|\triangle^k \hat{\psi}\|_\infty \right)$, where $\triangle = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$ is the frequency domain Laplacian operator.

Proof. Observe that, for $t = \binom{t_1}{t_2}$ and $x = \binom{x_1}{x_2}$ in $\mathbb{R}^2$, we have:

$$\psi_{ast}(x) = |\det M|^{-\frac{1}{2}} \, \psi(M^{-1}(x-t)) = a^{-\frac{3}{4}} \, \psi \begin{pmatrix} a^{-1}\,(x_1 - t_1) + s\,a^{-1}(x_2 - t_2) \\ a^{-\frac{1}{2}}(x_2 - t_2) \end{pmatrix}.$$

The proof then follows from Proposition 2.2, where

$$R = \{(\xi_1, \xi_2) : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}], \; |s - \tfrac{\xi_2}{\xi_1}| \leq \sqrt{a}\}.$$

It is easy to check that $m(R) = \frac{15}{2} \frac{\sqrt{a}+s}{a^2}$. $\square$

4. Analysis of singularities

As observed above, the mother shearlet ψ, constructed in Section 3, satisfies $\hat{\psi} \in C_0^\infty(\widehat{\mathbb{R}}^2)$. It follows that $\psi \in \mathcal{S}(\mathbb{R}^2)$ and, therefore, the continuous shearlet transform $\mathcal{SH}_f(a,s,t) = \langle f, \psi_{ast} \rangle$, $a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2$, is well defined for all tempered distributions $f \in \mathcal{S}'$.

In the following, we will examine the behavior of the continuous shearlet transform of several distributions containing different types of singularities. This will be useful to illustrate the basic properties of the shearlet transform, before stating a more general result in the next section. Indeed, the rate of decay of the continuous shearlet transform exactly describes the location and orientation of the singularities.

In order to state our results, it will be useful to introduce the following notation to distinguish between the following two different behaviors of the continuous shearlet transform.

Definition 4.1. Let f be a distribution on $\mathbb{R}^2$, $\mathcal{SH}_f(a,s,t)$ be defined as in Definition 3.3, and let $r \in \mathbb{R}$. Then $\mathcal{SH}_f(a,s,t)$ *decays rapidly as* $a \to 0$, if

$$\mathcal{SH}_f(a,s,t) = O(a^k) \quad \text{as } a \to 0, \text{ for every } k \geq 0.$$

We use the notation: $\mathcal{SH}_f(a,s,t) \sim a^r$ *as* $a \to 0$, if there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha a^r \leq \mathcal{SH}_f(a,s,t) \leq \beta a^r \quad \text{as } a \to 0.$$

4.1. **Point singularities.** We start by examining the decay properties of the continuous shearlet transform of the Dirac δ.

Proposition 4.2. *If $t = 0$, we have*

$$\mathcal{SH}_\delta(a,s,t) \sim a^{-\frac{3}{4}} \quad \text{as } a \to 0.$$

In all other cases, $\mathcal{SH}_\delta(a,s,t)$ decays rapidly as $a \to 0$.

Proof. For $t = 0$ we have

$$\langle \delta, \psi_{ast} \rangle = \psi_{as0}(0) = a^{-\frac{3}{4}}\, \psi(0) \sim a^{-\frac{3}{4}} \ \text{ as } a \to 0.$$

Next let $t \neq 0$. Then

$$\langle \delta, \psi_{ast} \rangle = \psi_{ast}(0),$$

and, by Proposition 3.4, for each $k \in \mathbb{N}$, we have

$$|\psi_{ast}(0)| \le C_k\, a^{-\frac{3}{4}} (1 + a^{-2}t_1^2 + 2a^{-2}st_1t_2 + (1 + a^{-1}s^2)a^{-1}t_2^2)^{-k}.$$

Thus, if $t_2 \neq 0$, then $|\psi_{ast}(0)| = O(a^{k-3/4})$ as $a \to 0$. Otherwise, if $t_2 = 0$, $t_1 \neq 0$, then $|\psi_{ast}(0)| = O(a^{2k-3/4})$ as $a \to 0$. □

4.2. **Linear singularities.** Next we will consider the linear delta distribution $\nu_p(x_1, x_2) = \delta(x_1 + p\,x_2)$, $p \in \mathbb{R}$, defined by

$$\langle \nu_p, f \rangle = \int_{\mathbb{R}} f(-p\,x_2, x_2)\, dx_2.$$

The following result shows that the continuous shearlet transform precisely determines both the position and the orientation of the linear singularity, in the sense that the transform $\mathcal{SH}_{\nu_p}(a,s,t)$ always decays rapidly as $a \to 0$ *except* when t is on the singularity and $s = p$, i.e., the direction perpendicular to the singularity.

Proposition 4.3. *If $t_1 = -p\,t_2$ and $s = p$, we have*

$$\mathcal{SH}_{\nu_p}(a,s,t) \sim a^{-\frac{1}{4}} \quad \text{as } a \to 0.$$

In all other cases, $\mathcal{SH}_{\nu_p}(a,s,t)$ decays rapidly as $a \to 0$.

Proof. The following heuristic argument gives

$$\begin{aligned}
\hat{\nu}_p(\xi_1,\xi_2) &= \int\!\!\int \delta(x_1 + p\,x_2)\, e^{-2\pi i \xi x}\, dx_2\, dx_1 \\
&= \int e^{-2\pi i x_2(\xi_2 - p\,\xi_1)}\, dx_2 = \delta(\xi_2 - p\,\xi_1) = \nu_{(-\frac{1}{p})}(\xi_1,\xi_2).
\end{aligned}$$

That is, the Fourier transform of the linear delta on $\mathbb{R}^2$ is another linear delta on $\widehat{\mathbb{R}}^2$, where the slope $-\frac{1}{p}$ is replaced by the slope p. A direct computation gives:

$$\begin{aligned}
\langle \hat{\nu}_p\,, \hat{\psi}_{ast} \rangle &= \int_{\mathbb{R}} \hat{\psi}_{ast}(\xi_1, p\xi_1)\, d\xi_1 \\
&= a^{\frac{3}{4}} \int_{\mathbb{R}} \hat{\psi}(a\xi_1, \sqrt{a}p\xi_1 - \sqrt{a}s\xi_1)\, e^{-2\pi i \xi_1(t_1 + pt_2)}\, d\xi_1 \\
&= a^{-\frac{1}{4}} \int_{\mathbb{R}} \hat{\psi}(\xi_1, a^{-\frac{1}{2}}p\xi_1 - a^{-\frac{1}{2}}s\xi_1)\, e^{-2\pi i a^{-1}\xi_1(t_1 + pt_2)}\, d\xi_1 \\
&= a^{-\frac{1}{4}} \int_{\mathbb{R}} \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(p - s))\, e^{-2\pi i a^{-1}\xi_1(t_1 + pt_2)}\, d\xi_1
\end{aligned}$$

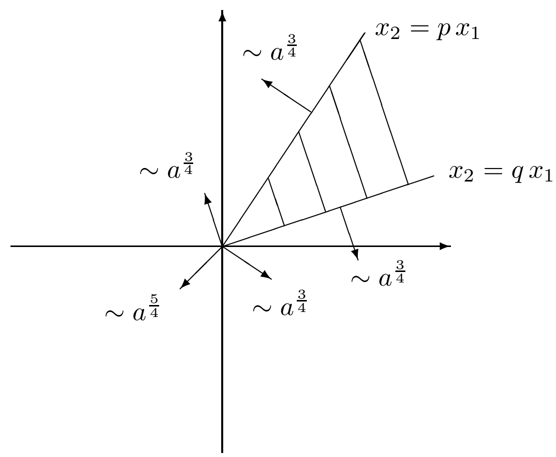

FIGURE 2. Decay properties of the continuous shearlet transform $\mathcal{SH}_{\chi_V}(a,s,t)$.

$$= \quad a^{-\frac{1}{4}}\,\hat{\psi}_2(a^{-\frac{1}{2}}(p-s))\,\psi_1(-a^{-1}(t_1+pt_2)).$$

If $s \neq p$, then there exists some $a>0$ such that $|p-s| > \sqrt{a}$. This implies that $\hat{\psi}_2(a^{-1/2}(p-s)) = 0$, and so $\langle \hat{\nu}_p\,, \hat{\psi}_{ast}\rangle = 0$. On the other hand, if $t_1 = -p\,t_2$ and $s=p$, then $\hat{\psi}_2(a^{-1/2}(p-s)) = \hat{\psi}_2(0) \neq 0$, and

$$\langle \hat{\nu}_p\,, \hat{\psi}_{ast}\rangle = a^{-\frac{1}{4}}\,\hat{\psi}_2(a^{-\frac{1}{2}}(p-s))\,\psi_1(0) \sim a^{-\frac{1}{4}} \quad \text{as } a \to 0.$$

If $t_1 \neq -p\,t_2$, by Proposition 2.2, we observe that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\langle \hat{\nu}_p\,, \hat{\psi}_{ast}\rangle \\
&\quad \leq \quad a^{-\frac{1}{4}}\,\hat{\psi}_2(a^{-\frac{1}{2}}(p-s))\,|\psi_1(a^{-1}(t_1+pt_2))| \\
&\quad \leq \quad C_k\,a^{-\frac{1}{4}}\,\hat{\psi}_2(a^{-\frac{1}{2}}(p-s))\,(1+a^{-2}(t_1+pt_2)^2)^{-k} = O(a^{2k-\frac{1}{4}}) \text{ as } a \to 0. \quad \square
\end{aligned}$$

4.3. **Polygonal singularities.** Here we consider the characteristic function χ_V of the cone $V = \{(x_1,x_2) : x_1 \geq 0,\ qx_1 \leq x_2 \leq px_1\}$, where $0 < q \leq p < \infty$. We have the following result.

Proposition 4.4. *For $t=0$, if $s = -\frac{1}{p}$ or $s = -\frac{1}{q}$, we have*

$$\mathcal{SH}_{\chi_V}(a,s,t) \sim a^{\frac{3}{4}} \quad \text{as } a \to 0,$$

and if $s \neq -\frac{1}{p}$ and $s \neq -\frac{1}{q}$, we have

$$\mathcal{SH}_{\chi_V}(a,s,t) \sim a^{\frac{5}{4}} \quad \text{as } a \to 0.$$

For $t \neq 0$, if $s = -\frac{1}{p}$ or $s = -\frac{1}{q}$, we have

$$\mathcal{SH}_{\chi_V}(a,s,t) \sim a^{\frac{3}{4}} \quad \text{as } a \to 0.$$

In all other cases, $\mathcal{SH}_{\chi_V}(a,s,t)$ decays rapidly as $a \to 0$.

The decay of the continuous shearlet transform of χ_V is illustrated in Figure 2. As shown in the figure, the decay of $\mathcal{SH}_{\chi_V}(a,s,t)$ exactly identifies the location and orientation of the singularities. It is interesting to notice that the orientation of the linear singularities can even be detected considering only the 'point singularity' at the origin.

Proof of Proposition 4.4. The Fourier transform of χ_V can be computed to be

$$\hat{\chi}_V(\xi_1,\xi_2) = C\frac{1}{(\xi_1+q\xi_2)(\xi_1+p\xi_2)}, \quad \text{where } C = \frac{(p+q)^2}{(2\pi)^2}.$$

A direct computation gives:

$$\begin{aligned}
&\langle \hat{\chi}_V, \hat{\psi}_{ast}\rangle \\
&= Ca^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{(\xi_1+q\xi_2)(\xi_1+p\xi_2)}\,\hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1}-s))\,e^{2\pi i\xi t}\,d\xi_1\,d\xi_2 \\
&= Ca^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{(a^{-1}\xi_1+q\xi_2)(a^{-1}\xi_1+p\xi_2)}\,\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(a\tfrac{\xi_2}{\xi_1}-s)) \\
&\quad\cdot e^{2\pi i(a^{-1}\xi_1,\xi_2)t}\,d\xi_1\,d\xi_2 \\
&= Ca^{-\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(a^{-1}\xi_1+q\xi_1(a^{-1/2}\xi_2+a^{-1}s))(a^{-1}\xi_1+p\xi_1(a^{-1/2}\xi_2+a^{-1}s))} \\
&\quad\cdot\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\xi_2)\,e^{2\pi i\xi_1(a^{-1}(t_1+st_2)+a^{-1/2}\xi_2t_2)}\,d\xi_1\,d\xi_2 \\
&= Ca^{\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(a^{-1/2}\xi_1(1+sq)+q\xi_1\xi_2)(a^{-1/2}\xi_1(1+sp)+p\xi_1\xi_2)} \\
&\quad\cdot\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\xi_2)\,e^{2\pi i\xi_1(a^{-1}(t_1+st_2)+a^{-1/2}\xi_2t_2)}\,d\xi_1\,d\xi_2.
\end{aligned}$$

Let us first consider the case $t=0$. By the previous computation we can rewrite $\langle \hat{\chi}_V, \hat{\psi}_{as0}\rangle$ as

$$Ca^{\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(a^{-1/2}\xi_1(1+sq)+q\xi_1\xi_2)(a^{-1/2}\xi_1(1+sp)+p\xi_1\xi_2)}\,\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\xi_2)\,d\xi_1\,d\xi_2.$$

If $s \neq -\frac{1}{p}$ and $s \neq -\frac{1}{q}$, for $a \ll 1$ we can rewrite $\langle \hat{\chi}_V, \hat{\psi}_{as0}\rangle$ as

$$\begin{aligned}
&Ca^{\frac{5}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(\xi_1(1+sq)+a^{1/2}q\xi_1\xi_2)(\xi_1(1+sp)+a^{1/2}p\xi_1\xi_2)}\,\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\xi_2)\,d\xi_1\,d\xi_2 \\
&\quad\sim\ C'a^{\frac{5}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(\xi_1(1+sq))(\xi_1(1+sp))}\,\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\xi_2)\,d\xi_1\,d\xi_2,
\end{aligned}$$

hence

$$\langle \hat{\chi}_V, \hat{\psi}_{as0}\rangle \sim a^{\frac{5}{4}} \quad \text{as } a \to 0.$$

The above computation also shows that if $t=0$ and $s=-\frac{1}{p}$ or $s=-\frac{1}{q}$, we have

$$\langle \hat{\chi}_V, \hat{\psi}_{as0}\rangle \sim a^{\frac{3}{4}} \quad \text{as } a \to 0.$$

Next, let us consider the situation, where t lies on one singularity but $t \neq 0$, i.e., $t_2 = pt_1$ or $t_2 = qt_1$. Here we will only examine the first case. The second one can be treated similarly. First let $s = -\frac{1}{p}$, i.e., s is perpendicular to the linear boundary of the cone $x_2 = px_1$. For $a \ll 1$ we have

$$\begin{aligned}
\langle \hat{\chi}_V, \hat{\psi}_{ast}\rangle &= Ca^{\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(a^{-1/2}\xi_1(1-q/p)+q\xi_1\xi_2)p\xi_1\xi_2}\hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2) \\
&\quad\cdot e^{2\pi i a^{-1/2}pt_1\xi_1\xi_2}d\xi_1 d\xi_2 \\
&= Ca^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(\xi_1(1-q/p)+a^{1/2}q\xi_1\xi_2)p\xi_1\xi_2} \\
&\quad\cdot\hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2)e^{2\pi i a^{-1/2}pt_1\xi_1\xi_2}d\xi_1 d\xi_2 \\
&\sim a^{\frac{3}{4}} \quad \text{as } a \to 0.
\end{aligned}$$

Secondly, let $s \neq -\frac{1}{p}$. We have:

$$\begin{aligned}
&\langle \hat\chi_V, \hat\psi_{ast}\rangle \\
&= Ca^{\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1}{(a^{-1/2}\xi_1(1+sq)+q\xi_1\xi_2)(a^{-1/2}\xi_1(1+sp)+a^{-1/2}p\xi_1\xi_2)} \\
&\quad \cdot \hat\psi_1(\xi_1)\,\hat\psi_2(\xi_2)\,e^{2\pi i\xi_1 t_1(a^{-1}(1+sp)+a^{-1/2}p\xi_2)}\,d\xi_1\,d\xi_2 \\
&= Ca^{\frac{1}{4}} \int_{\mathbb{R}} \varphi(\xi_1)\,\hat\psi_1(\xi_1)\,e^{2\pi i a^{-1}t_1(1+sp)\xi_1}\,d\xi_1,
\end{aligned}$$

where

$$\begin{aligned}
\varphi(\xi_1) &= \int_{\mathbb{R}} \frac{\xi_1}{(a^{-1/2}\xi_1(1+sq)+q\xi_1\xi_2)(a^{-1/2}\xi_1(1+sp)+p\xi_1\xi_2)}\hat\psi_2(\xi_2) \\
&\quad \cdot e^{2\pi i a^{-1/2}t_1 p\xi_1\xi_2} d\xi_2.
\end{aligned}$$

Since ψ_1 and ψ_2 are band-limited, the function φ has compact support, hence $(\varphi\hat\psi_1)^\vee$ is of rapid decay towards infinity. Thus

$$\langle \hat\chi_V, \hat\psi_{ast}\rangle = C\,a^{\frac{1}{4}}(\varphi\hat\psi_1)^\vee(a^{-1}t_1(1-sp)) = O(a^k) \quad \text{as } a\to 0.$$

Finally, in case $t_2 \neq pt_1$, $t_2 \neq qt_1$ and $t_1 \neq 0$, a similar argument to the one above shows that $\langle \hat\chi_V, \hat\psi_{ast}\rangle$ decays rapidly also in this case. $\square$

4.4. **Curvilinear singularities.** We will now examine the behavior of the continuous shearlet transform of a distribution having a discontinuity along a curve.

Let $B(x_1,x_2) = \chi_D(x_1,x_2)$, where $D = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 \le 1\}$. We have the following:

Proposition 4.5. *If $t_1^2+t_2^2=1$ and $s=\frac{t_2}{t_1}$, $t_1\neq 0$, we have*

$$\mathcal{SH}_B(a,s,t) \sim a^{\frac{3}{4}} \quad \text{as } a\to 0.$$

In all other cases, $\mathcal{SH}_B(a,s,t)$ decays rapidly as $a\to 0$.

The assumption $t_1 \neq 0$ shows that the shearlet transform $\mathcal{SH}_B(a,s,t)$ is unable to handle the vertical direction $s\to\infty$. To provide a complete analysis of the singularities of B, we need to use both $\mathcal{SH}_B(a,s,t)$ and $\mathcal{SH}_B^{(v)}(a,s,t)$ (as defined in Section 3). Since the shearlets $\psi_{ast}^{(v)}$ are defined on the vertical cone $C^{(v)}$, using $\mathcal{SH}_B^{(v)}(a,s,t)$ one can obtain a similar result to Proposition 4.5, for $s=\frac{t_1}{t_2}$, $t_2\neq 0$. Since the argument for both cases is exactly the same, we will only examine the transform $\mathcal{SH}_B(a,s,t)$.

In order to prove Proposition 4.5, we need to recall the following facts. First, we recall the asymptotic behavior of Bessel functions, that is given by the following lemma (cf. [24]):

Lemma 4.6. *There are constants C_1, C_2 such that*

$$J_1(2\pi\lambda) \sim \lambda^{-\frac{1}{2}}\,(C_1\,e^{2\pi i\lambda}+C_2\,e^{-2\pi i\lambda}) \quad \text{as } \lambda\to\infty,$$

and, for $N=1,2,\ldots$, constants C_1^N, C_2^N such that

$$\left(\frac{d}{d\lambda}\right)^N J_1(2\pi\lambda) \sim \lambda^{-\frac{1}{2}}\,(C_1^N\,e^{2\pi i\lambda}+C_2^N\,e^{-2\pi i\lambda}) \quad \text{as } \lambda\to\infty.$$

Secondly, we recall the following fact concerning oscillatory integrals of the First Kind, that can be found in [24, Ch.8]:

Lemma 4.7. *Let $A \in C_0^\infty(\mathbb{R})$ and $\Phi \in C^1(\mathbb{R})$, with $\Phi'(t) \neq 0$ on supp A. Then*

$$I(\lambda) = \int_{\mathbb{R}} A(t)\, e^{2\pi i \lambda \Phi(t)} dt = \frac{(-1)^N}{(2\pi i \lambda)^N} \int_{\mathbb{R}} D^N\big(A(t)\big)\, e^{2\pi i \lambda \Phi(t)} dt,$$

for $N = 1, 2, \dots,$ where $D\big(A(t)\big) = \frac{d}{dt}\big(\frac{A(t)}{\Phi'(t)}\big)$.

We can now prove Proposition 4.5.

Proof of Proposition 4.5. The continuous shearlet transform of $B(x)$ is given by:

(4.1)
$$\mathcal{SH}_B(a, s, t) = \langle B, \psi_{ast} \rangle = a^{\frac{3}{4}} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat\psi_1(a\xi_1)\, \hat\psi_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1} - s))\, e^{-2\pi i \xi t}\, \hat B(\xi)\, d\xi_1\, d\xi_2.$$

The Fourier transform $\hat B(\xi_1, \xi_2)$ is the radial function:

$$\hat B(\xi_1, \xi_2) = 2 \int_{-1}^{1} \sqrt{1 - x^2}\, e^{2\pi i \sqrt{\xi_1^2 + \xi_2^2} x}\, dx = |\xi|^{-1}\, J_1(2\pi |\xi|),$$

where J_1 is the Bessel function of order 1. Therefore, the asymptotic behavior of $\hat B(\lambda)$ follows from Lemma 4.6, with the factor $\lambda^{-1/2}$ replaced by $\lambda^{-3/2}$.

Because of the radial symmetry, it is convenient to convert (4.1) into polar coordinates:

$$\begin{aligned} &\mathcal{SH}_B(a, s, t) \\ &\quad = a^{\frac{3}{4}} \iint \hat\psi_1(a\rho\cos\theta)\, \hat\psi_2(a^{-\frac{1}{2}}(\tan\theta - s))\, e^{-2\pi i \rho(t_1 \cos\theta + t_2 \sin\theta)}\, \hat B(\rho)\, \rho\, d\rho\, d\theta \\ &\quad = a^{-\frac{5}{4}} \iint \hat\psi_1(\rho\cos\theta)\, \hat\psi_2(a^{-\frac{1}{2}}(\tan\theta - s))\, e^{-2\pi i \frac{\rho}{a}(t_1 \cos\theta + t_2 \sin\theta)}\, \hat B(\tfrac{\rho}{a})\, \rho\, d\rho\, d\theta. \qquad (4.2) \end{aligned}$$

We will now examine the asymptotic decay of the function $\mathcal{SH}_B(a, s, t)$ along the curve ∂B for $a \to 0$. Thus, we set $t_1^2 + t_2^2 = 1$ and, without loss of generality, assume $a < 1$. As we will show, the decay will depend on whether the direction associated with s is normal to the curve ∂B or not.

Let us begin by considering the non-normal case $s \neq t_2/t_1$. From (4.2), we have:

$$\mathcal{SH}_B(a, s, t) = a^{-\frac{5}{4}} \int I(a, \rho)\, \hat B(\tfrac{\rho}{a})\, \rho\, d\rho,$$

where (using the conditions on the support of $\hat\psi_2$)

$$I(a, \rho) = \int_{|\tan\theta - s| < \sqrt{a}} \hat\psi_1(\rho\cos\theta)\, \hat\psi_2(a^{-\frac{1}{2}}(\tan\theta - s))\, e^{-2\pi i \frac{\rho}{a}(t_1 \cos\theta + t_2 \sin\theta)}\, d\theta.$$

Observe that the domain of integration is the cone $|\tan\theta - s| < \sqrt{a}$ about the direction $\tan\theta = s$, with $a < 1$. This implies that θ ranges over an interval. Since the conditions on the support of $\hat\psi_1$ implies that $|\rho\cos\theta| \subset [\frac{1}{2}, 2]$, it follows that ρ also ranges over an interval and, as a consequence, $I(a, \rho)$ is compactly supported in ρ.

We will show that $I(a, \rho)$ is an oscillatory integral of the First Kind that decays rapidly for $a \to 0$ for each ρ. To show that this is the case, we will apply Lemma 4.7 to $I(a, \rho)$, where $A(\theta; \rho) = \hat\psi_1(\rho\cos\theta)\, \hat\psi_2(a^{-1/2}(\tan\theta - s))$, $\Phi(\theta; \rho) = \rho(t_1 \cos\theta + t_2 \sin\theta)$ and $\lambda = a^{-1}$ and ρ is a fixed parameter. Observe that $\Phi'(\theta; \rho) = \rho(-t_1 \sin\theta + t_2 \cos\theta)$ and $\Phi'(\theta; \rho) \neq 0$ for $\tan\theta \neq \frac{t_2}{t_1}$. Thus, we have that, if

$|s - \frac{t_2}{t_1}| \geq \sqrt{a}$, then the function $\Phi'(\theta;\rho) \neq 0$ on $\operatorname{supp} A$. A direct computation gives

$$\begin{aligned}
D\big(A(\theta;\rho)\big) &= \frac{\partial}{\partial\theta}\frac{\hat\psi_1(\rho\cos\theta)\,\hat\psi_2(a^{-\frac12}(\tan\theta - s))}{\rho(-t_1\sin\theta + t_2\cos\theta)} = \\
&= \frac{\sin\theta}{t_1\sin\theta - t_2\cos\theta}\hat\psi_1{}'(\rho\cos\theta)\,\hat\psi_2(a^{-\frac12}(\tan\theta - s)) + \\
&\quad + a^{-\frac12}\frac{\sec^2\theta}{\rho(t_2\cos\theta - t_1\sin\theta)}\hat\psi_1(\rho\cos\theta)\,\hat\psi_2'(a^{-\frac12}(\tan\theta - s)) + \\
&\quad + \frac{t_2\sin\theta + t_1\cos\theta}{\rho^2(t_2\cos\theta - t_1\sin\theta)^2}\,\hat\psi_1(\rho\cos\theta)\hat\psi_2(a^{-\frac12}(\tan\theta - s)).
\end{aligned}$$

Thus, using the assumptions on $\hat\psi_1, \hat\psi_2$, we have:

$$\big|D\big(A(\theta;\rho)\big)\big| < a^{-\frac12}\,C(\theta,\rho)\,\big(\|\hat\psi_1'\hat\psi_2\|_\infty + \|\hat\psi_1\hat\psi_2'\|_\infty + \|\hat\psi_1\hat\psi_2\|_\infty\big).$$

As observed above, the assumptions on the support of $\hat\psi_1, \hat\psi_2$ imply that $D\big(A(\theta;\rho)\big)$ is compactly supported in ρ away from $\rho = 0$. Using this observation and $\Phi'(\theta) \neq 0$, we have that

$$\|D(A)\|_\infty < C\,a^{-\frac12}.$$

Applying the same estimate repeatedly, we have

$$\|D^N(A)\|_\infty < C_N\,a^{-\frac{N}{2}}.$$

Thus, using Lemma 4.7 with $\lambda = a^{-1}$, we conclude that

$$\sup_\rho |I(a,\rho)| < C\,a^{\frac{N}{2}}.$$

This implies that, under the assumption that we made for $t = (t_1, t_2)$ and s, the function $\mathcal{SH}_B(a,s,t)$ decays rapidly for $a \to 0$.

Let us now consider the function $|\langle \hat B, \hat\psi_{ast}\rangle|$, where $t_1^2 + t_2^2 = 1$ and $s = t_2/t_1$ (corresponding to the direction normal to ∂B). For simplicity, let $(t_1,t_2) = (1,0)$. The general case follows using a similar argument. From (4.2), using the change of variables $u = a^{-1/2}\sin\theta$, we obtain

$$\langle \hat B, \hat\psi_{a0(1,0)}\rangle = a^{-\frac34}\int \hat B(\tfrac{\rho}{a})\,\eta_a(\rho)\;e^{2\pi i\frac{\rho}{a}}\rho\,d\rho \tag{4.3}$$

where

$$\eta_a(\rho) = \int_{-(1+a)^{-1/2}}^{(1+a)^{-1/2}} \hat\psi_1(\rho\sqrt{1-au^2})\,\hat\psi_2(\frac{u}{\sqrt{1-au^2}})\,e^{2\pi i\frac{\rho}{a}(\sqrt{1-au^2}-1)}\,\frac{du}{\sqrt{1-au^2}}.$$

The assumptions on the support of $\hat\psi_2$ imply that $|\frac{u}{\sqrt{1-au^2}}| < 1$. This is equivalent to $|u| < (1+a)^{1/2}$. Similarly, the assumptions on the support of $\hat\psi_1$ imply that $|\rho\sqrt{1-au^2}| \subset [\frac12, 2]$ and, thus, ρ ranges over a closed interval. As a consequence, the functions $\eta_a(\rho)$ are compactly supported. For $0 < a < 1$, the functions

$$h_a(u) = \hat\psi_1(\rho\sqrt{1-au^2})\,\hat\psi_2(\frac{u}{\sqrt{1-au^2}})\,\frac{e^{2\pi i\frac{\rho}{a}(\sqrt{1-au^2}-1)}}{\sqrt{1-au^2}}$$

are equicontinuous and they converge uniformly:

$$\lim_{a\to 0} h_a(u) = h_0(u) = \hat\psi_1(\rho)\,\hat\psi_2(u)\,e^{-\pi i\rho u^2}.$$

Thus, we have the uniform limit:

$$\lim_{a\to 0} \eta_a(\rho) = \eta_0(\rho) = \int_{-1}^{1} \hat{\psi}_1(\rho)\, \hat{\psi}_2(u)\, e^{-\pi i \rho u^2}\, d\rho,$$

and the same convergence holds for all u-derivatives. In particular, $\|\eta_a\|_\infty < C$, for all $a < 1$.

Using the asymptotic estimate given by Lemma 4.6 into (4.3), we have:

$$\begin{aligned} |\langle \hat{B}, \hat{\psi}_{a0(1,0)}\rangle| &\sim a^{-\frac{3}{4}} \Big(C_1 \int \Big(\frac{a}{\rho}\Big)^{\frac{3}{2}} \eta_a(\rho)\, e^{4\pi i \frac{\rho}{a}}\, \rho\, d\rho + C_2 \int \Big(\frac{a}{\rho}\Big)^{\frac{3}{2}} \eta_a(\rho)\, \rho\, d\rho \Big) \\ &= a^{\frac{3}{4}} \Big(C_1\, \hat{F}_a\Big(-\frac{2}{a}\Big) + C_2 \int F_a(\rho)\, d\rho \Big), \end{aligned}$$

where $F_a(\rho) = \eta_a(\rho)\, \rho^{-1/2}$. The family of functions $\{F_a : 0 < a < 1\}$ has all its ρ derivatives bounded uniformly in a, and so $\hat{F}_a(-\frac{2}{a})$ decays rapidly as $a \to 0$. On the other hand, $\int F_a(\rho)\, d\rho$ tends to $\int \eta_0(\rho)\, \rho^{-1/2}\, d\rho$, and thus, combining the two terms, we obtain that

$$|\langle \hat{B}, \hat{\psi}_{a0(1,0)}\rangle| \sim a^{\frac{3}{4}} \quad \text{as } a \to 0.$$

If t is not on ∂B, then one can show that $\mathcal{SH}_B(a,s,t)$ has fast decay. This follows from the general analysis given in Section 5. $\square$

5. Characterization of the wavefront set using the shearlet transform

The examples described in Section 4 suggest that the set of singularities of a distribution on $\mathbb{R}^2$ can be characterized using the continuous shearlet transform. In this section, we will show that this is indeed the case. In order to do this, it will be useful to introduce the notions of singular support and wavefront set.

For a distribution u, we say that $x \in \mathbb{R}^2$ is a *regular point* of u if there is $\phi \in C_0^\infty(U_x)$, where U_x is a neighborhood of x and $\phi(x) \neq 0$, such that $\phi\, u \in C_0^\infty(\mathbb{R}^n)$. Recall that the condition $\phi\, u \in C_0^\infty$ is equivalent to $(\phi\, u)^\wedge$ being rapidly decreasing. The complement of the regular points of u is called the *singular support* of u and is denoted by $\operatorname{sing\,supp}(u)$. It is easy to see that the singular support of u is a closed subset of $\operatorname{supp}(u)$.

The wavefront set of u consists of certain $(x,\lambda) \in \mathbb{R}^2 \times \mathbb{R}$, with $x \in \operatorname{sing\,supp}(u)$. For a distribution u, a point $(x,\lambda) \in \mathbb{R}^2 \times \mathbb{R}$ is a *regular directed point* for u if there are neighborhoods U_x of x and V_λ of λ, and a function $\phi \in C_0^\infty(\mathbb{R}^2)$, with $\phi = 1$ on U_x, so that, for each $N > 0$ there is a constant C_N with

$$|(u\,\phi)^\wedge(\eta)| \le C_N\, (1 + |\eta|)^{-N},$$

for all $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ with $\frac{\eta_2}{\eta_1} \in V_\lambda$. The complement in $\mathbb{R}^2 \times \mathbb{R}$ of the regular directed points for u is called the *wavefront set* of u and is denoted by $WF(u)$. Thus, the singular support is measuring the location of the singularities and λ is measuring the direction perpendicular to the singularity.[1]

In the examples presented in Section 4, one can verify the following:

[1] This definition is consistent with [6], where the direction of the singularity is described by the angle θ. Observe that our approach does not distinguish between θ and $\theta+\pi$, since the continuous shearlets have frequency support that is symmetric with respect to the origin. However, in Section 6 we discuss a variant of the continuous shearlet transform, which can distinguish these cases.

(i) *Point Singularity* $\delta(x)$*:* $\operatorname{sing\,supp}(\delta) = \{0\}$ and $WF(\delta) = \{0\} \times \mathbb{R}$.
(ii) *Linear Singularity* $\nu_p(x)$*:* $\operatorname{sing\,supp}(\nu_p) = \{(-px_2, x_2) : x_2 \in \mathbb{R}\}$ and $WF(\nu_p) = \{((-px_2, x_2), p) : x_2 \in \mathbb{R}\}$.
(iii) *Curvilinear Singularity* $B(x)$*:* $\operatorname{sing\,supp}(B) = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ and $WF(B) = \{((x_1, x_2), \lambda) : x_1^2 + x_2^2 = 1, \lambda = \frac{x_2}{x_1}\}$.

As observed in Section 4, all these sets are exactly identified by the decay properties of the continuous shearlet transform. Indeed, we have the following general result:

Theorem 5.1. (i) *Let* $\mathcal{R} = \{t_0 \in \mathbb{R}^2$ *: for* t *in a neighborhood* U *of* t_0*,* $|\mathcal{SH}_f(a, s, t)| = O(a^k)$ *and* $|\mathcal{SH}_f^{(v)}(a, s, t)| = O(a^k)$ *as* $a \to 0$*, for all* $k \in \mathbb{N}$*, with the* $O(\cdot)$*–terms uniform over* $(s, t) \in [-1, 1] \times U\}$*. Then*

$$\textit{sing supp}(f)^c = \mathcal{R}.$$

(ii) *Let* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$*, where* $\mathcal{D}_1 = \{(t_0, s_0) \in \mathbb{R}^2 \times [-1, 1]$ *: for* (s, t) *in a neighborhood* U *of* (s_0, t_0)*,* $|\mathcal{SH}_f(a, s, t)| = O(a^k)$ *as* $a \to 0$*, for all* $k \in \mathbb{N}$*, with the* $O(\cdot)$*–term uniform over* $(s, t) \in U\}$ *and* $\mathcal{D}_2 = \{(t_0, s_0) \in \mathbb{R}^2 \times [1, \infty)$ *: for* $(\frac{1}{s}, t)$ *in a neighborhood* U *of* (s_0, t_0)*,* $|\mathcal{SH}_f^{(v)}(a, s, t)| = O(a^k)$ *as* $a \to 0$*, for all* $k \in \mathbb{N}$*, with the* $O(\cdot)$*–term uniform over* $(\frac{1}{s}, t) \in U\}$*. Then*

$$\textit{WF}(f)^c = \mathcal{D}.$$

The statement (ii) of the theorem shows that the continuous shearlet transform $\mathcal{SH}_f(a, s, t)$ identifies the wavefront set for directions s such that $|s| = |\frac{\xi_2}{\xi_1}| \le 1$ (in the frequency domain). The continuous shearlet transform $\mathcal{SH}_f^{(v)}(a, s, t)$ identifies the wavefront set for directions s such that $|s| = |\frac{\xi_1}{\xi_2}| \le 1$, corresponding to $|\frac{\xi_2}{\xi_1}| \ge 1$. The proof of Theorem 5.1 will require several lemmata and will adapt several ideas from [6]. The following lemma shows that if t is outside the support of a function g, then the continuous shearlet transform of g decays rapidly as $a \to 0$.

Lemma 5.2. *Let* $g \in L^2(\mathbb{R}^2)$ *with* $\|g\|_\infty < \infty$*. If* $\textit{supp}\,(g) \subset \mathcal{B} \subset \mathbb{R}^2$*, then for all* $k > 1$*,*

$$|\mathcal{SH}_g(a, s, t)| = |\langle g, \psi_{ast}\rangle| \le C_k\, C(s)^2\, \|g\|_\infty\, a^{\frac{1}{4}} \left(1 + C(s)^{-1} a^{-1} d(t, \mathcal{B})^2\right)^{-k},$$

where $C(s) = \left(1 + \frac{s^2}{2} + \left(s^2 + \frac{s^4}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$ *and* C_k *is as in Proposition 3.4.*

Proof. Since $\|g\|_\infty < \infty$, by Proposition 3.4, for all $k \in \mathbb{N}$, there is a $C_k > 0$ such that:

$$\begin{aligned} |\langle g, \psi_{ast}\rangle| &\le \|g\|_\infty \int_{\mathcal{B}} |\psi_{ast}(x)| dx \\ &\le C_k\, \|g\|_\infty\, a^{-\frac{3}{4}} \int_{\mathcal{B}} \left(1 + \|M^{-1}(x - t)\|^2\right)^{-k} dx \\ &= C_k\, \|g\|_\infty\, a^{-\frac{3}{4}} \int_{\mathcal{B}+t} \left(1 + \|M^{-1}x\|^2\right)^{-k} dx, \end{aligned} \tag{5.1}$$

where $M = \begin{pmatrix} a & -\sqrt{a}\,s \\ 0 & \sqrt{a} \end{pmatrix}$. Observe that $\|x\| = \|MM^{-1}x\| \le \|M\|_{op}\, \|M^{-1}x\|$, and, thus,

$$\|M^{-1}x\| \ge \frac{1}{\|M\|_{op}}\, \|x\|.$$

Since $M = \left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & \sqrt{a} \end{smallmatrix}\right)$ and $\| \left(\begin{smallmatrix} a & 0 \\ 0 & \sqrt{a} \end{smallmatrix}\right) \|_{op} = \sqrt{a}$, then

$$\|M^{-1}x\| \geq C(s)^{-1}\, a^{-\frac{1}{2}}\, \|x\|,$$

where $C(s) = \| \left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right) \|_{op}$. Using these observation in (5.1), we have that

$$\begin{aligned} |\langle g, \psi_{ast}\rangle| &\leq C_k\, \|g\|_\infty\, a^{-\frac{3}{4}} \int_{\mathcal{B}+t} \left(1 + C(s)^{-2} a^{-1} \|x\|^2\right)^{-k} dx \\ &= C_k\, C(s)^2\, \|g\|_\infty\, a^{\frac{1}{4}} \int_{C(s)^{-1}a^{-1/2} d(t,\mathcal{B})}^{\infty} (1+r^2)^{-k} r\, dr \\ &= C_k\, C(s)^2\, \|g\|_\infty\, a^{\frac{1}{4}} \left(1 + C(s)^{-2} a^{-1} d(t,\mathcal{B})^2\right)^{-k}. \end{aligned}$$

At last, we compute $C(s)$. We have $\left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -s & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1+s^2 & -s \\ -s & 1 \end{smallmatrix}\right)$. The largest eigenvalue of this matrix is $\lambda_{\max} = 1 + \frac{s^2}{2} + \left(s^2 + \frac{s^4}{4}\right)^{\frac{1}{2}}$. Thus we have $C(s) = \left(1 + \frac{s^2}{2} + \left(s^2 + \frac{s^4}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$ for all $s \in \mathbb{R}$. $\square$

We can now prove the following inclusions.

Proposition 5.3. *Let $\mathcal{R}$ and $\mathcal{D}$ be defined as in Theorem 5.1. Then:*

(i) *sing supp$(f)^c \subseteq \mathcal{R}$.*
(ii) *$WF(f)^c \subseteq \mathcal{D}$.*

Proof. (i) Let t_0 be a regular point of f. Then there exists $\phi \in C_0^\infty(\mathbb{R}^2)$ with $\phi(t_0) \equiv 1$ on $B(t_0, \delta)$, which is the ball centered at t_0 with radius δ, such that $\phi f \in C^\infty(\mathbb{R}^2)$. We will show that $t_0 \in \mathcal{R}$. For this, we decompose $\mathcal{SH}_f(a,s,t)$ as

$$\mathcal{SH}_f(a,s,t) = \langle \psi_{ast}, \phi f\rangle + \langle \psi_{ast}, (1-\phi) f\rangle. \tag{5.2}$$

Observe that we have

$$|\langle \psi_{ast}, \phi f\rangle| \leq a^{\frac{3}{4}} \int_{\mathbb{R}^2} |\hat{\psi}_1(a\xi_1)|\, |\hat{\psi}_2(\tfrac{1}{\sqrt{a}}(\tfrac{\xi_2}{\xi_1} - s))|\, |\widehat{\phi f}(\xi)|\, d\xi.$$

In the following, we will estimate the above integral for $\xi_1 > 0$. The case $\xi_1 \leq 0$ is similar.

Since $\phi \in C_0^\infty(\mathbb{R}^2)$, for each $k \in \mathbb{N}$ there exists a constant C_k with $|\widehat{\phi f}(\xi)| \leq C_k\, |\xi|^{-2k}$. Using this fact, together the assumptions on the support of $\hat{\psi}_{ast}$, for $k > 2$, the first term on the RHS of (5.2) can be estimated as follows:

$$\begin{aligned} & a^{\frac{3}{4}} \int_{\mathbb{R}^+ \times \mathbb{R}} |\hat{\psi}_1(a\xi_1)|\, |\hat{\psi}_2(\tfrac{1}{\sqrt{a}}(\tfrac{\xi_2}{\xi_1} - s))|\, |\widehat{\phi f}(\xi)|\, d\xi \\ \leq\; & C_k \|\hat{\psi}\|_\infty a^{\frac{3}{4}} \int_{\frac{1}{2a}}^{\frac{2}{a}} \int_{(s-\sqrt{a})\xi_1}^{(s+\sqrt{a})\xi_1} |\xi|^{-2k}\, d\xi_2\, d\xi_1 \\ \leq\; & C_k 2^{-k} \|\hat{\psi}\|_\infty a^{\frac{3}{4}} \int_{\frac{1}{2a}}^{\frac{2}{a}} \xi_1^{-k} \int_{(s-\sqrt{a})\xi_1}^{(s+\sqrt{a})\xi_1} \xi_2^{-k}\, d\xi_2\, d\xi_1 \\ =\; & \frac{C_k 2^{-k} \|\hat{\psi}\|_\infty a^{\frac{3}{4}}}{1-k} \left((s+\sqrt{a})^{1-k} - (s-\sqrt{a})^{1-k}\right) \int_{\frac{1}{2a}}^{\frac{2}{a}} \xi_1^{1-2k}\, d\xi_1 \\ =\; & \frac{C_k 2^{-k} \|\hat{\psi}\|_\infty a^{\frac{3}{4}}}{1-k} \left((\sqrt{a}-s)^{1-k} - (-\sqrt{a}-s)^{1-k}\right) \frac{1}{1-2k} \left((\tfrac{2}{a})^{2-2k} - (\tfrac{1}{2a})^{2-2k}\right) \end{aligned}$$

$$\leq \quad \frac{C_k 2^{-k} \|\hat{\psi}\|_\infty a^{\frac{3}{4}}}{k(2k-1)} (\sqrt{a} - s)^{1-k} (\tfrac{2}{a})^{2-2k}.$$

Thus, the above quantity is a $O(a^k)$ as $a \to 0$, uniformly over $(t,s) \in B(t_0, \frac{\delta}{2}) \times \mathbb{R}$.

Using Lemma 5.2, we have the following estimate for the second term on the RHS of (5.2):

$$|\langle \psi_{ast}, (1-\phi) f\rangle| \leq C_k\, C(s)^2\, \|(1-\phi)f\|_\infty\, a^{\frac{1}{4}}\, (1 + C(s)^{-1} a^{-1} d(t, B(t_0,\delta)^c)^2)^{-k},$$

where $k \in \mathbb{N}$ is arbitrary. Since $\|(1-\phi)f\|_\infty < \infty$ and s is bounded, this yields

$$|\langle \psi_{ast}, (1-\phi) f\rangle| = O(a^k) \quad \text{as } a \to 0,$$

uniformly over $(t,s) \in B(t_0, \frac{\delta}{2}) \times [-1,1]$. A similar estimate hold when $\mathcal{SH}_f(a,s,t)$ is replaced by $\mathcal{SH}_f^{(v)}(a,s,t)$. This proves (i).

(ii) Let (t_0, s_0) be a regular directed point of f, with $s_0 \in [-1,1]$. Then there exists a $\phi \in C_0^\infty(\mathbb{R}^2)$ with $\phi(t_0) \equiv 1$ on a ball $B(t_0, \delta_1)$ such that, for each $k \in \mathbb{N}$ we have $|\widehat{\phi f}(\xi)| = O((1+|\xi|)^{-k})$ for all $\xi \in \widehat{\mathbb{R}}^2$ satisfying $\frac{\xi_2}{\xi_1} \in B(s_0, \delta_2)$. We will prove that $(t_0, s_0) \in \mathcal{D}$. For this, we decompose $\mathcal{SH}_f(a,s,t)$ as in (5.2). The second term on the RHS of (5.2) can be estimated as in the case (i). For the first term of (5.2), we only need to show that $\operatorname{supp} \hat{\psi}_{ast} \subset \{\xi \in \mathbb{R}^2 : \frac{\xi_2}{\xi_1} \in B(s_0, \delta_2)\}$ for all $(s,t) \in B(s_0, \delta_2) \times B(t_0, \delta_1)$, since in this cone $\widehat{\phi f}$ decays rapidly. As above, we only consider the case $\xi_1 > 0$; the case $\xi_1 \leq 0$ is similar. The support of $\hat{\psi}_{ast}$ in this half plane is given by

$$\{(\xi_1, \xi_2) : \xi_1 \in [\tfrac{1}{2a}, \tfrac{2}{a}], \xi_2 \in \xi_1 [s - \sqrt{a}, s + \sqrt{a}]\}.$$

Let $(s,t) \in B(s_0, \delta_2) \times B(t_0, \delta_1)$. The cone $\{\xi \in \mathbb{R}^2 : \frac{\xi_2}{\xi_1} \in B(s_0, \delta_2)\}$ is bounded by the lines $\xi_2 = (s_0 - \delta_2)\xi_1$ and $\xi_2 = (s_0 + \delta_2)\xi_1$. Now let $(\xi_1, \xi_2) \in \operatorname{supp} \hat{\psi}_{ast}$. Then, for a small enough, we have

$$|\tfrac{\xi_2}{\xi_1} - s_0| \leq \sqrt{a} \leq \delta_2,$$

and this completes the proof in this case. In case $|s_0| \geq 1$ (this corresponds to $|\frac{\xi_2}{\xi_1}| \leq 1$), we proceed exactly as above, using the transform $\mathcal{SH}_f^{(v)}(a,s,t)$ rather than $\mathcal{SH}_f(a,s,t)$. $\square$

For the converse inclusions we need some additional lemmata. For simplicity of notation, in the following proofs the symbols C' and C_k are generic constants and may vary from expression to expression (in the case of C_k, the constant depends on k).

Lemma 5.4. *Let $S \subset \mathbb{R}$ be a compact set, and $g \in L^2(\mathbb{R})$ with $\|g\|_\infty < \infty$. Suppose that $\operatorname{supp} g \subset \mathcal{B}$ for some $\mathcal{B} \subset \mathbb{R}^2$ and define $(\mathcal{B}^\eta)^c = \{x \in \mathbb{R}^2 : d(x, \mathcal{B}) > \eta\}$. Further define $h \in L^2(\mathbb{R})$ by*

$$\hat{h}(\xi) = \int_0^\infty \int_{(\mathcal{B}^\eta)^c} \int_S \mathcal{SH}_g(a,s,t)\, \hat{\psi}_{ast}(\xi)\, ds\, dt\, \frac{da}{a^3}.$$

Then $\hat{h}(\xi)$ decays rapidly as $|\xi| \to \infty$ with constants dependent only on $\|g\|_\infty$ and η.

Proof. Using the fact that S is compact, Lemma 5.2 implies that, for each $k > 0$,

$$|\mathcal{SH}_g(a,s,t)| \leq C_k\, a^{\frac{1}{4}}\, (1 + a^{-1} d(t, \mathcal{B})^2)^{-k},$$

where C_k depends on $\|g\|_\infty$ but not on s. By definition, the support of $\hat{\psi}_{ast}$ is contained in the set

$$\Gamma(a,s) = \{\xi \in \mathbb{R}^2 : \tfrac{1}{2} \le a|\xi| \le 2,\ |s - \tfrac{\xi_2}{\xi_1}| \le \sqrt{a}\}. \tag{5.3}$$

Thus, $|\hat{\psi}_{ast}(\xi)| \le C' a^{\frac{3}{4}} \chi_{\Gamma(a,s)}(\xi)$ and

$$\int_S \chi_{\Gamma(a,s)}(\xi)\, ds \le \int_{S \cap \left[\frac{\xi_2}{\xi_1} - \sqrt{a}, \frac{\xi_2}{\xi_1} + \sqrt{a}\right]} ds \le C'\sqrt{a}. \tag{5.4}$$

Collecting the above arguments,

$$\begin{aligned}
\hat{h}(\xi) &\le \int_0^\infty \int_{(\mathcal{B}^\eta)^c} \int_S |\mathcal{SH}_g(a,s,t)|\, |\hat{\psi}_{ast}(\xi)|\, ds\, dt\, \frac{da}{a^3} \\
&\le C_k \int_0^\infty \int_{(\mathcal{B}^\eta)^c} \int_S a\, \chi_{\Gamma(a,s)}(\xi)\, (1 + a^{-1} d(t,\mathcal{B})^2)^{-k}\, ds\, dt\, \frac{da}{a^3} \\
&\le C_k \int_0^\infty \int_{(\mathcal{B}^\eta)^c} \int_S \chi_{\Gamma(a,s)}(\xi)\, ds\, a\, (1 + a^{-1} d(t,\mathcal{B})^2)^{-k}\, dt\, \frac{da}{a^3} \\
&\le C_k \int_{\frac{1}{2|\xi|}}^{\frac{2}{|\xi|}} a^{-\frac{3}{2}} \int_{(\mathcal{B}^\eta)^c} (1 + a^{-1} d(t,\mathcal{B})^2)^{-k}\, dt\, da \\
&\le C_k \int_{\frac{1}{2|\xi|}}^{\frac{2}{|\xi|}} a^{-\frac{3}{2}} \int_\eta^\infty (1 + a^{-1} r^2)^{-k}\, r\, dr\, da \\
&\le C_k \int_{\frac{1}{2|\xi|}}^{\frac{2}{|\xi|}} a^{-\frac{1}{2}}\, (1 + a^{-1}\eta^2)^{-k+2}\, da \\
&\le C_k\, |\xi|^{-\frac{1}{2}}\, (1 + |\xi|\, \eta^2)^{-k+2}.
\end{aligned}$$

Since this holds for each $k > 0$, $\hat{h}(\xi)$ decays rapidly as $|\xi| \to \infty$. □

Lemma 5.5. *Let $S \subset \mathbb{R}$ and $\mathcal{B} \subset \mathbb{R}^2$ be compact sets. Suppose that $G(a,s,t)$ decays rapidly as $a \to 0$ uniformly for $(s,t) \in S \times \mathcal{B}$. Define $h \in L^2(\mathbb{R})$ by*

$$\hat{h}(\xi) = \int_0^\infty \int_{\mathcal{B}} \int_S G(a,s,t)\, \hat{\psi}_{ast}(\xi)\, ds\, dt\, \frac{da}{a^3}.$$

Then $\hat{h}(\xi)$ decays rapidly as $|\xi| \to \infty$.

Proof. As in Lemma 5.4, we will use the fact that $|\hat{\psi}_{ast}(\xi)| \le C'\, a^{\frac{3}{4}} \chi_{\Gamma(a,s)}(\xi)$, where $\Gamma(a,s)$ is given by (5.3), and the estimate (5.4). Also, by hypothesis, for each $k > 0$ and $a > 0$ we have

$$\sup\{|G(a,s,t)| : |\xi| \in [\tfrac{1}{2a}, \tfrac{2}{a}],\ t \in \mathcal{B}\} \le C_k\, a^k.$$

Using all these observation, we have that, for each $k > 0$,

$$\begin{aligned}
|\hat{h}(\xi)| &\le \int_0^\infty \int_{\mathcal{B}} \int_S |G(a,s,t)|\, |\hat{\psi}_{ast}(\xi)|\, ds\, dt\, \frac{da}{a^3} \\
&\le C_k \int_0^\infty \int_{\mathcal{B}} \int_S \chi_{\Gamma(a,s)}(\xi)\, a^{k-\frac{9}{4}}\, ds\, dt\, da \\
&\le C_k \int_{\frac{1}{2|\xi|}}^{\frac{2}{|\xi|}} a^{k-\frac{7}{4}}\, da
\end{aligned}$$

$$\leq \quad C_k\, |\xi|^{-k+\frac{7}{4}}. \quad \square$$

The proof of the following lemma adapts several ideas from [3, Lemma 2.3].

Lemma 5.6. *Suppose* $0 \leq a_0 \leq a_1 < 1$ *and* $|s| \leq s_0$*. Then for* $K > 1$*, there is a constant* C_K*, dependent on* K *only, such that:*

$$|\langle \psi_{a_0 s t}, \psi_{a_1 s' t'} \rangle| \leq C_K \left(1 + \frac{a_1}{a_0}\right)^{-K} \left(1 + \frac{|s-s'|^2}{a_1}\right)^{-K} \left(1 + \frac{\|(t-t')\|^2}{a_1}\right)^{-K}$$

Proof. By the properties of ψ, we have that, for $\|\xi\| > \frac{1}{2}$ and any $k > 0$, there is a corresponding constant C_k such that

$$|\hat{\psi}(\xi)| \leq C_k \, \frac{1}{(1 + |\xi_1| + |\xi_2|)^k}.$$

We also have that $\hat{\psi}(\xi) = 0$ for $\|\xi\| < \frac{1}{2}$. Thus, observing that $M_{as}^t \xi = (a\, \xi_1, \sqrt{a}\, \xi_2 - \sqrt{a}\, s\, \xi_1)$, it follows that:

$$|\hat{\psi}_{ast}(\xi)| \leq C_k \, \frac{a^{\frac{3}{4}}}{(1 + a\,|\xi_1| + \sqrt{a}\,|\xi_2 - s\,\xi_1|)^k}.$$

Using polar coordinates, by writing $\xi_1 = r\,\cos\theta$ and $\xi_2 = r\,\sin\theta$, this expression can be written as

$$|\hat{\psi}_{ast}(r,\theta)| \leq C_k \, \frac{a^{\frac{3}{4}}}{(1 + a\, r|\cos\theta| + \sqrt{a}\, r|\sin\theta - s\, \cos\theta|)^k}.$$

For $|\theta| \leq \pi/2$, using the assumption $|s| \leq s_0$, the last expression can be controlled by

$$|\hat{\psi}_{ast}(r,\theta)| \leq C_k \, \frac{a^{\frac{3}{4}}}{\left(1 + a\, r + \sqrt{a}\, r\, |\sin\theta - s|\right)^k}. \tag{5.5}$$

In addition, since $\sin\theta \sim \theta$ on $|\theta| \leq \pi/2$, we can replace $\sin\theta$ with θ in the above expression.

Let $\Delta s = s - s'$, $a_0 = \min(a, a')$ and $a_1 = \max(a, a')$. Using (5.5), and applying the same argument on $|\theta| \leq \pi/2$ and $\pi/2 < |\theta| \leq \pi$, we have that

$$\begin{aligned}
&\int_{\widehat{\mathbb{R}}^2} \left| \hat{\psi}_{a's't'}(\xi)\, \hat{\psi}_{ast}(\xi) \right| d\xi \\
&\leq C_k \int_{\frac{1}{a_0}}^{\infty} \int_{-\pi}^{\pi} \frac{(a\,a')^{\frac{3}{4}}\, r}{(1 + a\, r + \sqrt{a}\, r\, |\theta - s|)^k \left(1 + a'\, r + \sqrt{a'}\, r\, |\theta - s'|\right)^k} \, d\theta\, dr \\
&\leq C_k \int_{a_0^{-1}}^{\infty} \frac{(a\,a')^{3/4} r}{(1 + a\, r)^{k'}\, (1 + a'\, r)^k} \int_{-\infty}^{\infty} \frac{1}{(1 + \alpha\,|\theta|)^k\, (1 + \alpha'\, |\theta + \Delta s|)^k} \, d\theta\, dr,
\end{aligned}$$

where $\alpha = \frac{\sqrt{a}\, r}{1 + a\, r}$, $\alpha' = \frac{\sqrt{a'}\, r}{1 + a'\, r}$. Using a simple calculation we have that, for $\alpha > \alpha' > 1$, $k > 1$,

$$\int_{-\infty}^{\infty} \frac{1}{(1 + \alpha\,|\theta|)^k\, (1 + \alpha'\, |\theta + \Delta s|)^k} \, d\theta \leq C_k \, \frac{1}{\alpha\, (1 + \alpha'\, |\Delta s|)^k}.$$

From the definition α', for $r \geq 1/a'$, we have that $\frac{1}{2}\frac{1}{\sqrt{a'}} \leq \alpha' \leq \frac{1}{\sqrt{a'}}$. Thus, for $r \geq 1/a'$, provided $k > 1$, the last expression gives

$$\int_{-\infty}^{\infty} \frac{1}{(1+\alpha\,|\theta|)^k\,(1+\alpha'\,|\theta+\Delta s|)^k}\,d\theta \leq C_k\,\frac{(1+a\,r)}{\sqrt{a}\,r}\,\left(1+\frac{|\Delta s|}{\sqrt{a'}}\right)^{-k}. \tag{5.6}$$

Another simple estimate, provided $k' > 1$, gives:

$$\int_{\frac{1}{a_0}}^{\infty} \frac{1}{(1+a_0\,r)^{k'}\,(1+a_1\,r)^k}\,dr \leq C_{k'}\,\frac{1}{a_0}\,\left(1+\frac{a_1}{a_0}\right)^{-k}. \tag{5.7}$$

Thus, using (5.6) and (5.7), for $\alpha > \alpha' > 1$, $a_0 = a'$ and $a_1 = a$, we have that

$$\begin{aligned}\int_{\widehat{\mathbb{R}}^2} \left|\hat\psi_{a's't'}(\xi)\,\hat\psi_{ast}(\xi)\right| d\xi \;&\leq\; C_k\,\left(\frac{a_1}{a_0}\right)^{\frac{1}{4}}\left(1+\frac{a_1}{a_0}\right)^{-k+1}\left(1+\frac{|\Delta s|}{\sqrt{a_0}}\right)^{-k}\\ &\leq\; C_k\,\left(1+\frac{a_1}{a_0}\right)^{-k+2}\left(1+\frac{|\Delta s|}{\sqrt{a_0}}\right)^{-k}.\end{aligned}$$

Similarly, for $\alpha > \alpha' > 1$, $a_1 = a'$ and $a_0 = a$, a similar calculation gives

$$\begin{aligned}\int_{\widehat{\mathbb{R}}^2} \left|\hat\psi_{a's't'}(\xi)\,\hat\psi_{ast}(\xi)\right| d\xi \;&\leq\; C_k\,\left(\frac{a_1}{a_0}\right)^{\frac{3}{4}}\left(1+\frac{a_1}{a_0}\right)^{-k}\left(1+\frac{|\Delta s|}{\sqrt{a_1}}\right)^{-k}\\ &\leq\; C_k\,\left(1+\frac{a_1}{a_0}\right)^{-k+1}\left(1+\frac{|\Delta s|}{\sqrt{a_1}}\right)^{-k}.\end{aligned}$$

In general, renaming the index k, we can show that

$$\int_{\widehat{\mathbb{R}}^2} \left|\hat\psi_{a's't'}(\xi)\,\hat\psi_{ast}(\xi)\right| d\xi \leq C_k\,\left(1+\frac{a_1}{a_0}\right)^{-k}\left(1+\frac{|\Delta s|}{\sqrt{a_1}}\right)^{-k} \tag{5.8}$$

We have:

$$\frac{\partial}{\partial\xi_1}\hat\psi_{ast}(\xi) = (a-\sqrt{a}\,s)\,\hat\psi_{ast}(\xi), \quad \frac{\partial}{\partial\xi_2}\hat\psi_{ast}(\xi) = \sqrt{a}\,\hat\psi_{ast}(\xi),$$

$$\frac{\partial^2}{\partial\xi_1^2}\hat\psi_{ast}(\xi) = (a-\sqrt{a}\,s)^2\,\hat\psi_{ast}(\xi), \quad \frac{\partial^2}{\partial\xi_2^2} = a\,\hat\psi_{ast}(\xi).$$

Thus, observing that $a, a' < 1$ and $|s| < s_0$, we have that

$$\left|\Delta_\xi\,\hat\psi_{ast}(\xi)\,\overline{\hat\psi_{a's't'}(\xi)}\right| \leq C'\,a_1\,|\hat\psi_{ast}(\xi)|\,|\hat\psi_{a's't'}(\xi)|.$$

Set

$$L = I - \frac{\Delta_\xi}{(2\pi)^2\,a_1}.$$

On the one hand, for each k, we have

$$\left|L^k\left(\hat\psi_{ast}\,\overline{\hat\psi_{a's't'}}\right)(\xi)\right| \leq C'\,|\hat\psi_{ast}(\xi)|\,|\hat\psi_{a's't'}(\xi)|.$$

On the other hand

$$L^k(e^{-2\pi i\xi(t-t')}) = \left(1+\frac{\|t-t'\|^2}{a_1}\right)^k\,e^{-2\pi i\xi(t-t')}. \tag{5.9}$$

Repeated integrations by parts give

$$\langle\psi_{ast},\psi_{a's't'}\rangle \;=\; \int_{\mathbb{R}^2}\hat\psi_{ast}(\xi)\,\overline{\hat\psi_{a's't'}(\xi)}\,d\xi$$

$$= \int_{\mathbb{R}^2} (a\,a')^{3/4}\,\hat{\psi}(M_{as}^t\xi)\,\overline{\hat{\psi}(M_{a's'}^t\xi)}\,e^{-2\pi i\xi(t-t')}\,d\xi$$
$$= \int_{\mathbb{R}^2} L^k\left((a\,a')^{3/4}\,\hat{\psi}(M_{as}^t\xi)\,\overline{\hat{\psi}(M_{a's'}^t\xi)}\right)\,L^{-k}\left(e^{-2\pi i\xi(t-t')}\right)\,d\xi.$$

Therefore, from the last expression, using (5.8)–(5.9), we have

$$|\langle\psi_{ast},\psi_{a's't'}\rangle| \le C_k\left(1+\frac{\|t-t'\|^2}{a_1}\right)^{-k}\left(1+\frac{a_1}{a_0}\right)^{-k}\left(1+\frac{|\Delta s|}{\sqrt{a_1}}\right)^{-k}.$$

The proof is completed recalling that, for $m > 0$, $(1+|x|)^{-2m} \sim (1+|x|^2)^{-m}$. That is, there are constants $C_1, C_2 > 0$ such that $C_1\,(1+|x|^2)^{-m} \le (1+|x|)^{-2m} \le C_2\,(1+|x|^2)^{-m}$.

From Lemma 5.6, the following result can be easily deduced.

Lemma 5.7. *Let $\phi_1 \in C^\infty(\mathbb{R}^2)$ supported in $B(0,1)$, and define $\phi(x) = \phi_1(a_\phi^{-1}(x-t))$.*

(i) *Suppose $0 \le \sqrt{a_0} \le \sqrt{a_1} \le a_\phi < 1$. Then for $K > 0$,*

$$|\langle\phi\,\psi_{a_0st},\psi_{a_1s't'}\rangle| \le C_K\left(1+\frac{a_1}{a_0}\right)^{-K}\left(1+\frac{|s-s'|^2}{a_1}\right)^{-K}\left(1+\frac{\|(t-t')\|^2}{a_1}\right)^{-K}.$$

(ii) *Suppose $0 \le \sqrt{a_0} \le a_\phi \le \sqrt{a_1} < 1$, $a_1 \le a_\phi$. Then for $K > 0$,*

$$|\langle\phi\,\psi_{a_0st},\psi_{a_1s't'}\rangle| \le C_K\left(1+\frac{a_1}{a_0}\right)^{-K}\left(1+\frac{|s-s'|^2}{a_\phi^2}\right)^{-K}\left(1+\frac{\|(t-t')\|^2}{a_1}\right)^{-K}.$$

(iii) *Suppose $0 \le \sqrt{a_0} \le a_\phi \le a_1 \le \sqrt{a_1} < 1$. Then for $K > 0$,*

$$|\langle\phi\,\psi_{a_0st},\psi_{a_1s't'}\rangle| \le C_K\left(1+\frac{a_\phi}{a_0}\right)^{-K}\left(1+\frac{\|(t-t')\|^2}{a_\phi^2}\right)^{-K}.$$

Now we now complete the proof of Theorem 5.1.

Proof of Theorem 5.1.

Since one direction was proved by Proposition 5.3, we only have to prove the inclusions:

(i) $\mathcal{R} \subseteq \text{sing supp}(f)^c$;
(ii) $\mathcal{D} \subseteq \text{WF}(f)^c$.

First we prove (i). Let $t_0 \in \mathcal{R}$. Then for all $t \in B(t_0,\delta)$, a ball centered at t_0, we have that $|\mathcal{SH}_f(a,s,t)| = O(a^k)$ as $a \to 0$, for all $k \in \mathbb{N}$ with the $O(\cdot)$–term uniform over $(t,s) \in B(t_0,\delta) \times [-1,1]$. A similar estimate holds for $\mathcal{SH}_f^{(v)}(a,s,t)$.

Choose $\phi \in C^\infty(\mathbb{R}^2)$ which is supported in a ball $B(t_0,\nu)$ with $\nu \ll \delta$ and let $\eta = \frac{\delta}{2}$. Set $g = \phi f$ and consider the decomposition

$$\widehat{\phi f}(\xi) = \hat{g}_0(\xi) + \hat{g}_1(\xi) + \hat{g}_2(\xi) + \hat{g}_3(\xi) + \hat{g}_4(\xi),$$

where $\hat{g}_0(\xi) = (\phi\,P(f))^\wedge(\xi)$, where $P(f) = \int_{\mathbb{R}} \langle f, T_b W\rangle\, T_b W\,db$, W is the window function defined by (3.6), and

$$\hat{g}_i(\xi) = \chi_{C_1}(\xi)\int_{\mathcal{Q}_i} \hat{\psi}_{ast}(\xi)\,\mathcal{SH}_g(a,s,t)\,d\mu(a,s,t), \quad i = 1,2,$$

$$\hat{g}_{i+2}(\xi) = \chi_{C_2}(\xi) \int_{\mathcal{Q}_i} \hat{\psi}_{ast}(\xi)\, \mathcal{SH}_g^{(v)}(a,s,t)\, d\mu(a,s,t), \quad i = 1, 2,$$

where C_1, C_2 are defined after equation (3.6), $d\mu(a,s,t) = \frac{da}{a^3}\, ds\, dt$, $\mathcal{Q}_1 = [0,1] \times [-2,2] \times B(t_0,\eta)$ and $\mathcal{Q}_2 = [0,1] \times [-2,2] \times B(t_0,\eta)^c$. The term $\hat{g}_0(\xi)$ decays rapidly as $|\xi| \to \infty$ since ϕ and $P(f)$ are in C^∞. The term $\hat{g}_2(\xi)$ decays rapidly as $|\xi| \to \infty$ by Lemma 5.4. In addition, by Lemma 5.5, $\hat{g}_1(\xi)$ decays rapidly as $|\xi| \to \infty$ provided that $\mathcal{SH}_g$ decays rapidly as $a \to 0$ uniformly over $(t,s) \in B(t_0,\eta) \times [-2,2]$. We will consider only the analysis of the terms $\hat{g}_i$, for $i = 1,2$; the cases $i = 3,4$ are very similar.

We claim that $\mathcal{SH}_g$ indeed decays rapidly as $a \to 0$ uniformly over $B(t_0,\eta) \times [-2,2]$. In order to prove this, we decompose f as $f = P(f) + P_{C_1} f + P_{C_2} f$, where $(P_{C_1} f)^\wedge = \hat{f}\, \chi_{C_1}$ and $(P_{C_2} f)^\wedge = \hat{f}\, \chi_{C_2}$. It is clear that $\mathcal{SH}_{\phi\, P(f)}$ decays rapidly by the smoothness of ϕ and $P(f)$. Next, we examine the term $P_{C_1} f$. The analysis of $P_{C_2} f$ is very similar and will be omitted. We use the decomposition $P_{C_1} f = f_1 + f_2$, where

$$f_i(x) = \int_{\mathcal{Q}_i} \psi_{ast}(x)\, \mathcal{SH}_f(a,s,t)\, d\mu(a,s,t), \quad i = 1, 2.$$

Let us start by considering the term corresponding to f_1. We have:

$$\mathcal{SH}_{\phi f_1}(a,s,t) = \langle \phi\, f_1, \psi_{ast} \rangle = \int_{\mathcal{Q}_1} \langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle\, \mathcal{SH}_f(a',s',t')\, d\mu(a',s',t').$$

We will decompose $\mathcal{Q}_1 = \mathcal{Q}_{10} \cup \mathcal{Q}_{11} \cup \mathcal{Q}_{12}$, corresponding to $a' > \delta$, $a' \le \delta < \sqrt{a'}$ and $\sqrt{a'} \le \delta$, respectively. In case $\sqrt{a}, \sqrt{a'} \le \delta$, by Lemma 5.7 we have that

$$|\langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle| \le C_K \left(1 + \frac{a_1}{a_0}\right)^{-K} \left(1 + \frac{\|(t-t')\|^2}{a_1}\right)^{-K}. \tag{5.10}$$

This implies that, for $m > 4$ and $K \ge m-1$

$$\int_0^\delta \left(1 + \frac{a_1}{a_0}\right)^{-K} (a')^m \, \frac{da'}{(a')^3} \le C_{m,K}\, a^{m-2}, \qquad 0 < a < \delta. \tag{5.11}$$

In fact, for $a' = a_0 \le a = a_1$,

$$\int_0^a \left(1 + \frac{a}{a'}\right)^{-K} (a')^m \, \frac{da'}{(a')^3} = a^{m-2} \int_0^1 (1+x)^{-K}\, dx = C_K\, a^{m-2}.$$

For $a = a_0 \le a' = a_1$,

$$\begin{aligned} \int_a^\delta \left(1 + \frac{a'}{a}\right)^{-K} (a')^m \, \frac{da'}{(a')^3} &= a^{m-2} \int_1^{\delta/a} x^{m-3}\, (1+x)^{-K}\, dx \\ &\le a^{m-2} \int_1^{\infty} x^{m-3}\, (1+x)^{-K}\, dx = C_{K,m}\, a^{m-2}. \end{aligned}$$

Thus, (5.11) follows from the last two estimates. Using (5.11) it follows that

$$\begin{aligned} &\int_{\mathcal{Q}_{12}} \langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle\, \mathcal{SH}_f(a',s',t')\, d\mu(a',s',t') \\ &\le C' \int_{-2}^{2} \int_{B(t_0,\eta)} \int_0^\delta \left(1 + \frac{a_1}{a_0}\right)^{-K} (a')^m \, \frac{da'}{(a')^3}\, dt'\, ds' \\ &\le C_m\, a^{m-2}, \end{aligned}$$

for all $m > 4$. Using the other cases of Lemma 5.7, one can show similar estimates for the integrals over the sets $\mathcal{Q}_{10}$ and $\mathcal{Q}_{11}$. This shows that $\mathcal{SH}_{\phi f_1}(a,s,t)$ decays rapidly for $a \to 0$ uniformly over $B(t_0,\eta) \times [-2,2]$.

Let us consider now the term corresponding to f_2:

$$\mathcal{SH}_{\phi f_2}(a,s,t) = \langle \phi\, f_2, \psi_{ast} \rangle = \int_{\mathcal{Q}_1} \langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle \, \mathcal{SH}_f(a',s',t')\, d\mu(a',s',t').$$

We will decompose $\mathcal{Q}_2 = \mathcal{Q}_{21} \cup \mathcal{Q}_{22}$, corresponding to $\|(t-t')\| > \eta$ and $\|(t-t')\| \le \eta$, respectively. Observe that, for $\|(t-t')\| > \eta$ and $K > 1$,

$$\int_{B(t_0,\eta)^c} \left(1 + \frac{\|(t-t')\|^2}{a_1}\right)^{-K} dt' \le \int_\eta^\infty \left(1 + \frac{r^2}{a_1}\right)^{-K} r\, dr \le C'\, a_1 \left(1 + \frac{\eta}{a_1}\right)^{-K+2}.$$

Observe that, on the region $\mathcal{Q}_{21}$, the function $\mathcal{SH}_f(a',s',t')$ is bounded by $C'\,(a')^{3/4}$ since f is bounded. Thus

$$\begin{aligned}
&\int_{\mathcal{Q}_{21}} \langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle \, \mathcal{SH}_f(a',s',t')\, d\mu(a',s',t') \\
&\le C' \int_{-2}^{2} \int_0^\delta \int_{B(t_0,\eta)^c} \left(1 + \frac{\|(t-t')\|^2}{a_1}\right)^{-K} dt' \left(1 + \frac{a_1}{a_0}\right)^{-K} (a')^{3/4}\, \frac{da'}{(a')^3}\, ds' \\
&\le C' \int_0^\eta a_1 \left(1 + \frac{\eta}{a_1}\right)^{-K+2} \left(1 + \frac{a_1}{a_0}\right)^{-K} \frac{da'}{(a')^{9/4}},
\end{aligned}$$

and this is of rapid decay, as $a \to 0$, uniformly over $\mathcal{Q}_{21}$. As for the region $\mathcal{Q}_{22}$, if $t \in B(t_0,\eta)$ and $\|(t-t')\| > \eta$, then $t' \in B(t_0,\delta)$ and thus the function $\mathcal{SH}_f$ decays rapidly, for $a \to 0$, over this region. Repeating the analysis as in the case $\mathcal{Q}_{12}$, we can prove that $\int_{\mathcal{Q}_{22}} \langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle \, \Psi_f(a',s',t')\, d\mu(a',s',t')$ is of rapid decay, as $a \to 0$, uniformly over $\mathcal{Q}_{22}$. Combining these observations, we conclude that $\mathcal{SH}_{\phi f_2}(a,s,t)$ decays rapidly as $a \to 0$ uniformly over $B(t_0,\eta) \times [-2,2]$.

It follows that $\mathcal{SH}_g(a,s,t)$ decays rapidly as $a \to 0$ uniformly for all $(s,t) \in B(t_0,\eta) \times [-2,2]$ and, thus, by Lemma 5.5, $\hat{g}_1(\xi)$ decays rapid as $|\xi| \to \infty$. We can now conclude that $\hat{g}$ decays rapidly as $|\xi| \to \infty$, hence completing the proof of (i).

For part (ii), we only sketch the idea of the proof, since it is very similar to part (i). Let $(t_0,s_0) \in \mathcal{D}$. We consider separately the case $|s_0| \le 1$ and $|s_0| \ge 1$. In the first case, for all $t \in B(t_0,\delta)$ and $s \in B(s_0,\Delta)$, we have that $|\mathcal{SH}_f(a,s,t)| = O(a^k)$, as $a \to 0$, for all $k \in \mathbb{N}$ with the $O(\cdot)$–term uniform over $(t,s) \in B(t_0,\delta) \times B(s_0,\Delta)$. Choose $\phi \in L^2(\mathbb{R}^2)$ which is supported in a ball $B(t_0,\nu)$ with $\nu \ll \delta$ and let $\eta = \frac{\delta}{2}$. Then the proof proceeds as in part (i), replacing $B(t_0,\delta) \times [-2,2]$ with $B(t_0,\delta) \times B(s_0,\Delta)$. Also, for the estimates involving inner products of ψ_{ast} and $\psi_{a's't'}$ we will now use Lemma 5.7 including the directionally sensitive term. For example, when $\sqrt{a}, \sqrt{a'} \le \delta$, by Lemma 5.7 we will use the estimate

$$|\langle \phi\, \psi_{ast}, \psi_{a's't'} \rangle| \le C_K \left(1 + \frac{a_1}{a_0}\right)^{-K} \left(1 + \frac{|s-s'|^2}{a_1}\right)^{-K} \left(1 + \frac{\|(t-t')\|^2}{a_1}\right)^{-K},$$

rather than (5.10). We can proceed similarly for the other estimates. The proof for the case $|s_0| \ge 1$ is exactly the same, with the transform $\mathcal{SH}_f^{(v)}(a,s,t)$ replacing $\mathcal{SH}_f(a,s,t)$. $\square$

6. Extensions and generalizations of the continuous shearlet transform

As mentioned above, there are several variants and generalizations of the continuous shearlets introduced in Section 3. In fact, using the theory of the affine systems, we can obtain several other examples of continuous wavelets depending on the three variables: scale, shear and location.

For example, we can generalize our construction by considering the case where Λ is given by (3.1) and M is of the form

$$M_\delta = \begin{pmatrix} a & -a^\delta\, s \\ & \\ 0 & a^\delta \end{pmatrix} = B\, A_\delta, \quad a \in I, s \in S,$$

where $B = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$, $A_\delta = \begin{pmatrix} a & 0 \\ 0 & a^\delta \end{pmatrix}$ and $0 \le \delta \le 1$ is fixed. If $\delta = \frac{1}{2}$, then we have the case of continuous shearlets. In general, for other choices of δ, A_δ will provide different kinds of anisotropic scaling. Using a construction similar to the continuous shearlets in Section 3, for each $0 \le \delta \le 1$, one can construct systems of the form

$$\{\psi_{ast} = T_t\, D_{M_\delta}\, \psi :\, (M_\delta, t) \in \Lambda\},$$

where $\hat{\psi}_{ast}$ is supported on the set:

$$\operatorname{supp} \hat{\psi}_{ast} \subset \{(\xi_1, \xi_2) : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}],\, |\tfrac{\xi_2}{\xi_1} - s| \le a^{1-\delta}\}.$$

This provides a new family of transforms $\mathcal{SH}^\delta_f(a, s, t) = \langle f, \psi_{ast}\rangle$, for various values of δ. It turns out that, provided $0 \le \delta < 1$, the behavior of the transforms $\mathcal{SH}^\delta_f(a, s, t)$ is very similar to the continuous shearlet transform in dealing with pointwise and linear singularities. More precisely, one can repeat the analysis in Sections 4.1, 4.2 and 4.3 using $\mathcal{SH}^\delta_f(a, s, t)$ (for $\delta \neq 1$) rather than the continuous shearlet transform. Similarly, the behavior of the transforms $\mathcal{SH}^\delta_f(a, s, t)$ is very similar to the continuous shearlet transform in dealing with curvilinear singularities (see Section 4.4), provided $0 < \delta < 1$. However, the proof of Theorem 5.1 requires $\delta = \frac{1}{2}$; that is, we need to use the continuous shearlet transform.

Another variant of affine systems generated by Λ, given by (3.1), is obtained by reversing the order of the shear and dilation matrices, namely, by letting

$$M_\delta = \begin{pmatrix} a & -a\, s \\ & \\ 0 & a^\delta \end{pmatrix} = A_\delta\, B, \quad a \in I, s \in S,$$

where A_δ, B are defined as above, and $0 \le \delta \le 1$ is fixed. Also in this case, we can construct variants of the continuous shearlets. Using the same ideas as above, we obtain a system of functions ψ_{ast} with support

$$\operatorname{supp} \hat{\psi}_{ast} \subset \{(\xi_1, \xi_2) : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}],\, |a^{1-\delta} \tfrac{\xi_2}{\xi_1} - s| \le a^{1-\delta}\}.$$

It turns out (as one can see from the support condition) that the transform associated with these systems is not even able to 'locate' the linear singularities, in the sense described in Section 4.2.

As we mentioned above, because the frequency support of the continuous shearlets is symmetric with respect to the origin, the continuous shearlet transform is unable to distinguish the orientation associated with the angle θ from the angle $\theta + \pi$. In order to be able to distinguish these two directions, we can modify our

construction as follows. Let $\psi \in L^2(\mathbb{R}^2)$ be defined as in Section 3, except that $\operatorname{supp}\hat{\psi}_1 \subset [\frac{1}{2}, 2]$. That is, we have a one-sided version of the shearlet $\hat{\psi}$ illustrated in Figure 1. It is then clear that, if we consider the affine system generated by ψ under the action of Λ given by (3.1) and (3.2), this cannot provide a reproducing system for all of $L^2(\mathbb{R}^2)$. In fact, ψ is a wavelet for the subspace $L^2(H)^\vee$, where $H = \{(\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \xi_1 \geq 0\}$. In order to obtain a wavelet for the space $L^2(\mathbb{R}^2)$, we can modify the set Λ as follows. Let Λ' given by (3.1), where $G \subset GL_2(\mathbb{R})$ is the set of matrices:

$$G = \left\{ M = \begin{pmatrix} \ell a & -\ell\sqrt{a}\, s \\ 0 & \ell\sqrt{a} \end{pmatrix}, \quad a \in I,\, s \in S,\, \ell = -1, 1 \right\}, \tag{6.1}$$

where $I \subset \mathbb{R}^+$, $S \subset \mathbb{R}$. Then the function ψ is a continuous wavelet for $L^2(\mathbb{R}^2)$ with respect to Λ'. These modified versions of the continuous shearlets depend on four variables a, s, t, ℓ and have frequency support:

$$\operatorname{supp}\hat{\psi}_{ast\ell} \subset \begin{cases} \{(\xi_1, \xi_2) : \xi_1 \in [-\frac{2}{a}, -\frac{1}{2a}],\, |\frac{\xi_2}{\xi_1} - s| \leq \sqrt{a}\}, & \text{if } \ell = -1, \\ \{(\xi_1, \xi_2) : \xi_1 \in [\frac{1}{2a}, \frac{2}{a}],\, |\frac{\xi_2}{\xi_1} - s| \leq \sqrt{a}\}, & \text{if } \ell = 1. \end{cases}$$

We remark that these modified versions of the continuous shearlets are in fact complex functions, whereas the shearlets we use throughout this paper are real functions.

Finally, there exists a natural way to construct continuous shearlets also in dimensions larger than 2. We refer to [15] for a discussion about the generalizations of shear matrices to higher dimensions.

Appendix A. Additional Computations

Proof of Theorem 2.1.

Suppose that (2.4) holds. Then, by applying Parseval and Plancherel formulas, for any $f \in L^2(\mathbb{R}^n)$ we have:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_G |\langle f, T_t\, D_M\, \psi\rangle|^2 \, d\lambda(M)\, dt \\
&= \int_{\mathbb{R}^n} \int_G \left| \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{\psi}(M^t\xi)}\, e^{2\pi i \xi t} d\xi \right|^2 |\det M|\, d\lambda(M)\, dt \\
&= \int_{\mathbb{R}^n} \int_G \left| \left(\hat{f}\, \overline{\hat{\psi}(M^t\cdot)} \right)^\vee (t) \right|^2 |\det M|\, d\lambda(M)\, dt \\
&= \int_G \int_{\mathbb{R}^n} \left| \left(\hat{f}\, \overline{\hat{\psi}(M^t\cdot)} \right)^\vee (t) \right|^2 dt\, |\det M|\, d\lambda(M) \\
&= \int_G \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, |\hat{\psi}(M^t\xi)|^2\, |\det M|\, d\xi\, d\lambda(M) \\
&= \int_G |\hat{f}(\xi)|^2\, \Delta(\psi)(\xi)\, d\xi = \|f\|^2.
\end{aligned}$$

Equation (2.3) follows from the above equality by polarization.

Conversely, suppose that

$$\int_{\mathbb{R}^n} \int_G |\langle f, T_t\, D_M\, \psi\rangle|^2 \, d\lambda(M)\, dt = \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$. Let ξ_0 be a point of differentiability of $\Delta(\psi)(\xi)$ and let $\hat{f}(\xi) = |B(\xi_0, r)|^{-1/2}\, \chi_{B(\xi_0,r)}(\xi)$, where $B(\xi_0, r)$ is a ball centered at ξ_0 of radius r. By reversing the chain of equalities above we conclude that

$$\frac{1}{|B(\xi_0, r)|} \int_{B(\xi_0,r)} \Delta(\psi)(\xi)\, d\xi = 1$$

for all $r > 0$. Letting $r \to 0$, we obtain that $\Delta(\psi)(\xi_0) = 1$, and, since a.e. $\xi \in \mathbb{R}^n$ is a point of differentiability, (2.4) holds. $\square$

The proof easily extends to function $f \in L^2(V)^\vee$. In fact, it suffices to replace $\hat{f}$ with $\hat{f}\chi_V$ in the argument above.

Proof of equality (3.7).

Using Plancherel and Parseval formulas we have that

$$\begin{aligned}
\int_{\mathbb{R}} |\langle f, T_t\, W\rangle|^2 \; dt &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{f}(\xi)\, \overline{\widehat{W}(\xi)}\, e^{2\pi i \xi t}\, d\xi \right|^2 \; dt \\
&= \int_{\mathbb{R}} \left| \left(\hat{f}\, \overline{\widehat{W}} \right)^\vee (t) \right|^2 \; dt \\
&= \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, |\widehat{W}(\xi)|^2 \; d\xi. \qquad \text{(A.1)}
\end{aligned}$$

Using a similar computation we have:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{-2}^{2} \int_0^1 |\langle P_{C_1} f, \psi_{ast}\rangle|^2 \; \frac{da}{a^3}\, ds\, dt \\
&= \int_{\mathbb{R}} \int_{-2}^{2} \int_0^1 \left| \int_{\mathbb{R}} \hat{f}(\xi)\, \chi_{C_1}(\xi)\, \overline{\hat{\psi}(M_{as}^t \xi)}\, e^{2\pi i \xi t}\, d\xi \right|^2 \; \frac{da}{a^{3/2}}\, ds\, dt \\
&= \int_{-2}^{2} \int_0^1 \int_{\mathbb{R}} \left| \left(\hat{f}\, \chi_{C_1}\, \overline{\hat{\psi}(M_{as}^t \cdot)} \right)^\vee (t) \right|^2 \; dt\, \frac{da}{a^{3/2}}\, ds \\
&= \int_{-2}^{2} \int_0^1 \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, \chi_{C_1}(\xi)\, |\hat{\psi}(M_{as}^t \xi)|^2\, d\xi\, \frac{da}{a^{3/2}}\, ds. \qquad \text{(A.2)}
\end{aligned}$$

As in the proof of Proposition 3.2, for $\xi \in C_1$, we have that

$$\int_{-2}^{2} \int_0^1 |\hat{\psi}(M_{as}^t \xi)|^2\, \frac{da}{a^{3/2}}\, ds = \int_0^1 |\hat{\psi}_1(a\xi_1)|^2\, \frac{da}{a}.$$

Thus, using the last equality, from (A.2) we have that

$$\int_{\mathbb{R}} \int_{-2}^{2} \int_0^1 |\langle P_{C_1} f, \psi_{ast}\rangle|^2 \; \frac{da}{a^3}\, ds\, dt = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \chi_{C_1}(\xi) \int_0^1 |\hat{\psi}_1(a\xi_1)|^2\, \frac{da}{a}\, d\xi. \qquad \text{(A.3)}$$

Similarly, we have that

$$\int_{\mathbb{R}} \int_{-2}^{2} \int_0^1 \left| \langle P_{C_2} f, \psi_{ast}^{(v)} \rangle \right|^2 \; \frac{da}{a^3}\, ds\, dt = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \chi_{C_2}(\xi) \int_0^1 |\hat{\psi}_1(a\xi_2)|^2\, \frac{da}{a}\, d\xi. \qquad \text{(A.4)}$$

Thus, combining (A.1), (A.3) and (A.4) and using (3.6) we have that

$$\begin{aligned}
&\int_{\mathbb{R}} |\langle f, T_t\, W\rangle|^2 \; dt + \int_{\mathbb{R}} \int_{-2}^{2} \int_0^1 |\langle P_{C_1} f, \psi_{ast}\rangle|^2 \; \frac{da}{a^3}\, ds\, dt \\
&+ \int_{\mathbb{R}} \int_{-2}^{2} \int_0^1 \left| \langle P_{C_2} f, \psi_{ast}^{(v)} \rangle \right|^2 \; \frac{da}{a^3}\, ds\, dt
\end{aligned}$$

$$= \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left(|\widehat{W}(\xi)|^2 + \chi_{C_1}(\xi) \int_0^1 |\hat{\psi}_1(a\xi_1)|^2 \, \frac{da}{a} + \chi_{C_2}(\xi) \int_0^1 |\hat{\psi}_1(a\xi_2)|^2 \, \frac{da}{a} \right) d\xi$$
$$= \|f\|^2. \quad \square$$

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Institute of Mathematics, Justus–Liebig–University Giessen, 35392 Giessen, Germany
E-mail address: `gitta.kutyniok@math.uni-giessen.de`

Department of Mathematics, North Carolina State University, Campus Box 8205, Raleigh, NC 27695, USA
E-mail address: `dlabate@unity.ncsu.edu`